

MARTINGALE PROBLEMS ON BANACH SPACES — EXISTENCE, UNIQUENESS AND THE MARKOV PROPERTY

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ABSTRACT. We prove under minimal assumptions on the coefficients that existence and uniqueness of solutions to semilinear stochastic evolution equations on a general separable Banach space E yields that the solutions are strong Markov processes.

We also establish existence and uniqueness (thus by the above also the strong Markov property) of solutions for stochastic reaction-diffusion equations with polynomially bounded nonlinearity and Hölder continuous multiplicative noise.

1. INTRODUCTION

In their seminal work [41], Stroock and Varadhan studied diffusion processes with very general coefficients. The main novelty in their approach was that they considered so-called ‘martingale problems’ rather than stochastic differential equations. Besides their very general existence and uniqueness results, it is their proof of the Markov property for diffusion processes that turned out to be most fruitful. It is the prototype of the following general principle:

Existence and uniqueness of solutions for the martingale problem with degenerate initial values imply that all solutions of the martingale problem are strong Markov processes. We call this the *Stroock-Varadhan principle*.

In this article, we establish a version of the Stroock-Varadhan principle for stochastic evolution equation

$$(1.1) \quad dX(t) = [AX(t) + F(X(t))]dt + G(X(t))dW_H(t) ,$$

on a separable Banach space E . Here, A is the generator of a strongly continuous semigroup \mathbf{S} on E , W_H is a cylindrical Wiener process with values in a separable Hilbert space H and the nonlinearities $F : E \rightarrow E$ and $G : E \rightarrow \mathcal{L}(H, E)$ satisfy suitable measurability and (local) boundedness assumptions. In fact, we shall consider even a more general situation to take into account smoothing effects of the semigroup. We will make our assumptions precise in Section 3.

Our second main result concerns the stochastic partial differential equation

$$(1.2) \quad \begin{aligned} \frac{\partial u}{\partial t}(t, x) = & [\operatorname{div} [a(x)\nabla u(t, x)] + f(x, u(t, x))] \\ & + \sum_{k=1}^{\infty} g_k(x, u(t, x)) \frac{\partial w_k}{\partial t}(t) \quad x \in \mathcal{O}, t > 0 \end{aligned}$$

on a domain $\mathcal{O} \subset \mathbb{R}^d$ with either Dirichlet or Neumann-type boundary conditions. Here $a \in C^{1,\alpha}(\mathcal{O}, \mathbb{R}^{d \times d})$ for some $\alpha > 0$, f is an odd-degree polynomial with continuous coefficients and strictly negative leading coefficient and the functions g_k are $\frac{1}{2}$ -Hölder continuous and of linear growth with suitable summability conditions.

2000 *Mathematics Subject Classification.* 60H15, 60J25.

Key words and phrases. Martingale solution, strong Markov property, stochastic partial differential equation, pathwise uniqueness, Feller property, elliptic operators.

The author was supported by VICI subsidy 639.033.604 in the ‘Vernieuwingsimpuls’ program of the Netherlands Organization for Scientific Research (NWO).

The sequence $(w_k)_{k \in \mathbb{N}}$ is a sequence of independent Brownian motions. We will prove existence and uniqueness of solutions for equation (1.2) and thus, by our Stroock-Varadhan principle, the strong Markov property for the solutions follows.

Let us compare our results with the existing literature.

In the case where E is a Hilbert space and F and G are Lipschitz continuous, the Markov property for solutions of (1.1) has been established by Da Prato and Zabczyk [10, Section 9] without referring to the martingale problem. Ondreját [35] has studied martingale problems (and established a version of the Stroock-Varadhan principle) on Banach spaces E which are 2-smoothable.

Both approaches make explicit use of an E -valued stochastic integral and this is also reflected in the (somewhat technical) assumptions on the coefficients A, F and G . There are two main motivations to extend the theory of martingale problems to general Banach spaces E .

First, it is desirable to have a version of the Stroock-Varadhan principle which does not depend on the stochastic integral used. Recent results on stochastic integration in Banach spaces [31, 8] make it possible to consider stochastic evolution equations in larger classes of Banach spaces. Thus, a Stroock-Varadhan principle which does not depend on the stochastic integral can be applied immediately to such equations, one does not have to check whether the proofs still work with the new stochastic integral.

Second, there are certain classes of stochastic equations (with stochastic reaction-diffusion equations (1.2) as a typical example) where the natural state-space E is a space of continuous functions such as $C(\overline{\mathcal{O}})$. Up to now, there is no satisfactory stochastic integration theory in such spaces. However, often stochastic integration in a smaller Banach space, typically a suitable Sobolev space, is sufficient to construct solutions. Even though the Sobolev space may be 2-smoothable, such equations do not fit into the framework of [35], at least as far as general initial values in E , rather than in the smaller Sobolev space are concerned.

We would like to point out a special difficulty in this case. Since the notion of a Markov process depends on the topology of the state space, it is not clear whether a Markov process in a Sobolev space, which is also a $C(\overline{\mathcal{O}})$ -valued stochastic process, is a Markov process in $C(\overline{\mathcal{O}})$. We should also note that in the case of strong Markov processes also the admissible stopping times depend on the state space.

In the present article, we base our theory on (analytically) weak, rather than mild solutions (as in [10, 35]), thereby avoiding the use of an E -valued stochastic integral altogether. This allows us to establish the Stroock-Varadhan principle (Theorem 3.8) on a general separable Banach space E , with no further assumptions on E . Also concerning the coefficients A, F and G our assumptions are minimal, in particular, we do not assume the semigroup \mathbf{S} to be analytic, nor do we assume G to take values in the space of all γ -radonifying operators. This makes our theory very flexible and thus easy to adapt to new notions of stochastic integration as well as to new methods of constructing solutions.

Let us now turn to the application of our theory to equation (1.2). We should first note that from a probabilistic point of view there are two different existence concepts, strong existence and (stochastically) weak existence. In the former, a probability space carrying a cylindrical Wiener process W_H is given and a solution is required to be defined on that given probability space. In the concept of a (stochastically) weak solution (in the infinite dimensional case, often the terminology *martingale solution* is used to distinguish stochastically weak solutions from analytically weak solutions) the probability space is part of the solution, i.e. we have to find some probability space carrying a cylindrical Wiener process, on which

a solution is defined. In the Stroock-Varadhan principle, existence refers to weak existence.

If the coefficients are Lipschitz continuous in a suitable sense, then strong existence of solutions can often be proved using Banach's fixed point theorem, see [10, 1, 32]. Existence of stochastically weak solutions can be proved for more general nonlinearities [42, 10, 2, 46].

Also concerning uniqueness of solutions, there are two different concepts. The uniqueness concept in the Stroock-Varadhan principle is that of *uniqueness in law*, i.e. any two solutions, defined on possibly different probability spaces, should have the same finite-dimensional distributions.

Pathwise uniqueness means that two solutions defined on the *same* probability space and with respect to the same Wiener process are indistinguishable. A classical result of Yamada and Watanabe [44] states that in the finite dimensional setting pathwise uniqueness implies uniqueness in law. This result generalizes to our infinite dimensional setting.

In the case where E is an infinite dimensional Hilbert space, uniqueness in law was studied by several authors [14, 7, 45]. We will discuss some cases of uniqueness in law for a general separable Banach space E elsewhere [22].

Pathwise uniqueness is often an immediate consequence of Banach's fixed point theorem and thus relatively standard for Lipschitz nonlinearities. Also if F and G are not (locally) Lipschitz continuous, pathwise uniqueness can sometimes be established under additional assumptions, see [43, 27].

Turning back to equation (1.2), existence of solutions was established under a uniform boundedness assumption on the g_k by Brzeźniak and Gątarek [2]. We extend this existence result to g_k of linear growth using some ideas of Cerrai [6], who studied equations with locally Lipschitz continuous coefficients (cf. also [23]). We note that Brzeźniak and Gątarek did not discuss uniqueness of solutions, except for the case of locally Lipschitz continuous coefficients.

In Section 5, we prove pathwise uniqueness for (1.2). This result is related to the article [27] by Mytnik, Perkins and Sturm who establish pathwise uniqueness for an equation involving the Laplacian on \mathbb{R}^d . We would like to point out that their proof depends on the translation invariance of the Laplace operator on \mathbb{R}^d and hence does not carry over immediately to our situation, where the differential operator has non-constant coefficients and is considered on a domain. Let us also mention Burdzy, Mueller and Perkins [4], who give an example with white noise, where pathwise uniqueness does not hold.

This article is organized as follows. In Section 2, we present some preliminary results about martingale problems on complete, separable metric spaces. In Section 3 we establish a one-to-one correspondence between weak solutions of stochastic evolution equation (1.1) and an (associated) martingale problem. Then the results of Section 2 are used to establish the Stroock-Varadhan principle for such equations (Theorem 3.8). In Section 4, we discuss the equivalence of (analytically) weak and mild solutions. In the concluding Section 5, we prove well-posedness of equation (1.2).

2. MARKOV PROCESSES AND LOCAL MARTINGALE PROBLEMS

In this section (E, d) is a complete, separable metric space. We denote the Borel σ -algebra of E by $\mathcal{B}(E)$. The spaces of scalar-valued measurable, bounded measurable, continuous and bounded continuous functions will be denoted by $B(E)$, $B_b(E)$, $C(E)$ and $C_b(E)$ respectively. $\mathcal{P}(E)$ denotes the set of all probability measures on $(E, \mathcal{B}(E))$. For $x \in E$, the Dirac measure in x is denoted by δ_x .

By $C([0, \infty); E)$ we denote the space of all continuous, E -valued functions. The elements of $(C[0, \infty); E)$ will be denoted by bold lower case letters: $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Endowed

with the metric δ , defined by

$$\delta(\mathbf{x}, \mathbf{y}) := \sum_{k=1}^{\infty} 2^{-k} \sup_{t \in [-k, k]} d(\mathbf{x}(t), \mathbf{y}(t)) \wedge 1,$$

$C([0, \infty); E)$ is a complete, separable metric space in its own right. We denote its Borel σ -algebra by \mathcal{B} . It is well-known that $\mathcal{B} = \sigma(\mathbf{x}(s) : s \geq 0)$, see [19, Lemma 16.1]. Here, in slight abuse of notation, we have identified $\mathbf{x}(s)$ with the E -valued random variable $\mathbf{x} \mapsto \mathbf{x}(s)$. We shall do so in what follows without further notice. The filtration generated by these ‘coordinate mappings’ is denoted by $\mathbb{B} := (\mathcal{B}_t)_{t \geq 0}$, i.e. $\mathcal{B}_t := \sigma(\mathbf{x}(s) : s \leq t)$.

The space $\mathcal{P}(C([0, \infty); E))$ will be topologized by the *weak topology*, i.e. the coarsest topology for which for all bounded continuous function Φ on $C([0, \infty); E)$, the map $\mathbf{P} \mapsto \int \Phi d\mathbf{P}$ is continuous. It is well known that this topology is metrizable through a complete, separable metric, see [38, Section II.6], i.e. $\mathcal{P}(C([0, \infty); E))$ is a Polish space.

A probability measure \mathbf{P} on $(C([0, \infty); E), \mathcal{B})$ is called a *Markov measure* if the coordinate process $(\mathbf{x}(t))_{t \geq 0}$ defined on $(C([0, \infty); E), \mathcal{B}, \mathbf{P})$ is a Markov process with respect to \mathbb{B} , i.e. for all $f \in B_b(E)$ and $s, t \geq 0$ we have

$$\mathbb{E}[f(\mathbf{x}(t+s)) | \mathcal{B}_t] = \mathbb{E}[f(\mathbf{x}(t+s)) | \mathbf{x}(t)] \quad \mathbf{P} - a.e.,$$

where \mathbb{E} denotes (conditional) expectation with respect to \mathbf{P} . If this equation also holds whenever t is replaced with a bounded \mathbb{B} -stopping time τ , i.e. the coordinate process is a strong Markov process with respect to \mathbb{B} , then \mathbf{P} is called a *strong Markov measure*.

A *transition semigroup* is a family $\mathbf{T} := (T(t))_{t \geq 0}$ of positive contractions on $B_b(E)$ such that

- (1) \mathbf{T} is a semigroup, i.e. $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.
- (2) Every operator $T(t)$ is associated with a *Markovian kernel*, i.e. a map $p_t : E \times \mathcal{B}(E) \rightarrow [0, 1]$ such that (i) $p_t(x, \cdot) \in \mathcal{P}(E)$ for all $x \in E$ and (ii) $p_t(\cdot, A) \in B_b(E)$ for all $A \in \mathcal{B}(E)$. That $T(t)$ is associated with p_t means that $T(t)f(x) = \int_E f(y) p_t(x, dy)$ for all $f \in B_b(E)$.

The kernels p_t themselves are referred to as *transition functions* or *transition probabilities*. The semigroup property above is equivalent with the *Chapman-Kolmogorov equations*.

A probability measure \mathbf{P} on $C([0, \infty); E)$ is called *Markov measure with transition semigroup* \mathbf{T} if for all $f \in B_b(E)$ and $s, t \geq 0$ we have

$$\mathbb{E}[f(\mathbf{x}(t+s)) | \mathcal{B}_t] = \mathbb{E}[f(\mathbf{x}(t+s)) | \mathbf{x}(t)] = [T(s)f](\mathbf{x}(t)) \quad \mathbf{P} - a.e.$$

If this equation also holds whenever t is replaced with a bounded \mathbb{B} -stopping time τ , then \mathbf{P} is called a *strong Markov measure with transition semigroup* \mathbf{T} .

The connection between martingale problems and Markovian measures is well established, see [11, Chapter 3]. However, in order to apply this theory to stochastic differential equations on Banach spaces, we have to consider *local* martingale problems rather than martingale problems.

Definition 2.1. An *admissible operator* is a map \mathcal{L} , defined on a subset $D(\mathcal{L}) \subset C(E)$ and taking values in $B(E)$ such that for all $f \in D(\mathcal{L})$ the function $\mathcal{L}f$ is bounded on compact subsets of E .

Given an admissible operator \mathcal{L} , a probability measure \mathbf{P} on $C([0, \infty); E)$ is said to *solve the local martingale problem for* \mathcal{L} if for every $f \in D(\mathcal{L})$ the process \mathbf{M}^f defined by

$$(\mathbf{M}^f(\mathbf{x}))(t) := f(\mathbf{x}(t)) - \int_0^t \mathcal{L}f(\mathbf{x}(s)) ds$$

is a local martingale under \mathbf{P} . This of course means that there exists a sequence τ_n , which may depend on f , of \mathbb{B} -stopping times with $\tau_n \uparrow \infty$ \mathbf{P} -almost surely such that the stopped processes $\mathbf{M}_{\tau_n}^f$, defined by $\mathbf{M}_{\tau_n}^f(t) := \mathbf{M}^f(t \wedge \tau_n)$, are martingales for all $n \in \mathbb{N}$.

If an initial distribution $\mu \in \mathcal{P}(E)$ is specified, we say that \mathbf{P} is a *solution to the local martingale problem for (\mathcal{L}, μ)* to indicate that in addition to being a solution to the local martingale problem for \mathcal{L} , the measure \mathbf{P} satisfies $\mathbf{P}(\mathbf{x}(0) \in \Gamma) = \mu(\Gamma)$ for all $\Gamma \in \mathcal{B}(E)$, i.e. under \mathbf{P} the random variable $\mathbf{x}(0)$ has distribution μ .

We note that by the continuity of $t \mapsto \mathbf{x}(t)$ and since $\mathcal{L}f$ is bounded on compact subsets of E , the process \mathbf{M}^f is well-defined. In fact, since f is a continuous function, it follows that \mathbf{M}^f is a continuous process. Before we proceed, we prove that the ‘localizing sequence’ τ_n in the above definition may be chosen independently of \mathbf{P} . We will, however, also have occasion to use other localizing sequences.

Lemma 2.2. *Let \mathcal{L} be admissible. For $f \in D(\mathcal{L})$, define $\tau_{n,f}$ by*

$$\tau_{n,f} := \inf \{t \geq 0; |\mathbf{M}^f(t)| \geq n\}.$$

Given a probability measure \mathbf{P} , the process \mathbf{M}^f is a local martingale under \mathbf{P} if and only if $\mathbf{M}_{\tau_{n,f}}^f$ is a martingale under \mathbf{P} for all $n \in \mathbb{N}$.

Proof. Since \mathbf{M}^f is a continuous process, the random time $\tau_{n,f}$ is a stopping time as hitting time of the closed set $\mathbb{R} \setminus (-n, n)$. Furthermore, $\tau_{n,f} \uparrow \infty$ pointwise and hence pointwise almost everywhere with respect to any probability measure \mathbf{P} on $C([0, \infty); E)$.

By [19, Lemma 17.1], \mathbf{M}^f is a local martingale under \mathbf{P} if and only if $\mathbf{M}_{\tau_{n,f}}^f$ is a local martingale under \mathbf{P} for all $n \in \mathbb{N}$. However, for fixed n , the process $\mathbf{M}_{\tau_{n,f}}^f$ is uniformly bounded by the definition of $\tau_{n,f}$. It follows from dominated convergence that $\mathbf{M}_{\tau_{n,f}}^f$ is a local martingale if and only if it is a (true) martingale. \square

The Stroock-Varadhan principle which we establish in the next section is based on the following theorem which is a straightforward generalization of [11, Theorem 4.4.2] to local martingale problems.

Theorem 2.3. *Let \mathcal{L} be admissible. Suppose that for every $\mu \in \mathcal{P}(E)$ any two solutions \mathbf{P}, \mathbf{Q} of the local martingale problem for (\mathcal{L}, μ) have the same one-dimensional distributions, i.e. for all $t \geq 0$ we have*

$$\mathbf{P}(\mathbf{x}(t) \in \Gamma) = \mathbf{Q}(\mathbf{x}(t) \in \Gamma) \quad \forall \Gamma \in \mathcal{B}(E).$$

Then

- (1) *Every solution of the local martingale problem for \mathcal{L} is a strong Markov measure.*
- (2) *For every $\mu \in \mathcal{P}(E)$, there is at most one solution to the local martingale problem for (\mathcal{L}, μ) .*

If in addition to the uniqueness assumption above for every $x \in E$ there exists a solution \mathbf{P}_x to the local martingale problem for (\mathcal{L}, δ_x) and if the map $x \mapsto \mathbf{P}_x(B)$ is Borel measurable for all $B \in \mathcal{B}$, then

- (3) *For every $\mu \in \mathcal{P}(E)$, there exists a solution \mathbf{P}_μ of the local martingale problem for (\mathcal{L}, μ) .*
- (4) *Define the operator $T(t)$ by $T(t)f(x) := \int f(\mathbf{x}(t)) d\mathbf{P}_x$ for $f \in B_b(E)$. Then every solution \mathbf{P} of the local martingale problem for \mathcal{L} is a strong Markov measure with transition semigroup $\mathbf{T} := (T(t))_{t \geq 0}$.*

It is well known that if \mathbf{P} is a Markov measure with transition semigroup \mathbf{T} , then \mathbf{T} and the distribution μ of $\mathbf{x}(0)$ under \mathbf{P} determine the finite-dimensional

distributions of \mathbf{P} and hence, since $\mathcal{B} = \sigma(\mathbf{x}(t) : t \geq 0)$, the measure \mathbf{P} uniquely. In fact, for $0 < t_1 < \dots < t_n < \infty$ and $A_j \in \mathcal{B}(E)$ for $0 \leq j \leq n$ we have

$$(2.1) \quad \begin{aligned} \mathbf{P}(\mathbf{x}(0) \in A_0, \dots, \mathbf{x}(t_n) \in A_n) &= \int_{A_0} \dots \int_{A_{n-1}} p_{t_n-t_{n-1}}(y_{n-1}, A_n) \\ &\quad \dots p_{t_{n-1}-t_{n-2}}(y_{n-2}, dy_{n-1}) \dots p_{t_1}(y_0, dy_1) d\mu(y_0). \end{aligned}$$

This proves that the additional measurability assumption in Theorem 2.3 is necessary for the existence of a semigroup which serves as a transition semigroup for *all* solutions of the local martingale problem. Indeed, (2.1) implies the measurability of $x \mapsto \mathbf{P}_x(B)$ for all cylinder sets B , and this extends to arbitrary $B \in \mathcal{B}$ by a monotone class argument. Let us also note that by [20, Theorem 17.24] the map $x \mapsto \mathbf{P}_x(B)$ is measurable for all $B \in \mathcal{B}$ if and only if the map $x \mapsto \mathbf{P}_x$ is Borel-measurable as a map from E to $\mathcal{P}(C([0, \infty); E))$.

Definition 2.4. Let \mathcal{L} be an admissible operator. We say that the local martingale problem for \mathcal{L} is *well-posed* if for every $x \in E$, there exists a unique solution \mathbf{P}_x of the local martingale problem for (\mathcal{L}, δ_x) .

We say that the martingale problem for \mathcal{L} is *completely well-posed*, if (i) for every $\mu \in \mathcal{P}(E)$ there exists a unique solution \mathbf{P}_μ of the local martingale problem for (\mathcal{L}, μ) and (ii) the map $x \mapsto \mathbf{P}_x(B)$ is measurable for every $B \in \mathcal{B}$.

In the case of uniqueness, we will use the notation \mathbf{P}_x resp. \mathbf{P}_μ for the solution of the local martingale problem for (\mathcal{L}, δ_x) , resp. (\mathcal{L}, μ) .

We note that by (2) of Theorem 2.3, the uniqueness assumption in the definition of ‘completely well-posed’ can be weakened to uniqueness of the one-dimensional marginals. Similarly, by (3) of Theorem 2.3, in the definition of ‘completely well-posed’ it suffices to assume existence of solutions only for degenerate initial distributions $\delta_x, x \in E$.

By part (4) of Theorem 2.3, if the local martingale problem for \mathcal{L} is completely well-posed, then there exists a transition semigroup \mathbf{T} such that every solution \mathbf{P}_μ is a strong Markov measure with transition semigroup \mathbf{T} . This semigroup \mathbf{T} is uniquely determined by \mathcal{L} and will be called the *associated semigroup*.

In the remainder of this section, we will discuss the *Feller property* of the semigroup \mathbf{T} . We recall that a transition semigroup \mathbf{T} is said to have the Feller property, if the space $C_b(E)$ is invariant under \mathbf{T} . In this case, it is interesting to study the continuity properties of the orbits $t \mapsto T(t)f$ for $f \in C_b(E)$. Here, two continuity concepts are of particular interest: *pointwise continuity* and *continuity with respect to the compact-open topology*. The latter means that for all $f \in C_b(E)$ we have $T(t)f \rightarrow T(s)f$ uniformly on the compact subsets of E as $t \rightarrow s$.

The special structure of the operators $T(t)$ as kernel operators allows us to use the theory of *semigroups on norming dual pairs* [24, 25] to study the semigroup \mathbf{T} . In the case of a Feller semigroup, we work on the norming dual pair $(C_b(E), \mathcal{M}(E))$, where $\mathcal{M}(E)$ denotes the space of all complex measures on $(E, \mathcal{B}(E))$.

In this framework, the continuity concepts above have a different interpretation. A transition semigroup with the Feller property is pointwise continuous if and only if the orbits $t \mapsto T(t)f$ are $\sigma(C_b(E), \mathcal{M}(E))$ -continuous for all $f \in C_b(E)$. Here, $\sigma(C_b(E), \mathcal{M}(E))$ is the *weak topology induced by $\mathcal{M}(E)$ on $C_b(E)$* . Thus, pointwise continuity corresponds to continuity with respect to the coarsest locally convex topology consistent with the duality.

On the other hand, continuity of the orbits with respect to the topology of uniform convergence on compact sets corresponds to continuity with respect to the finest locally convex topology consistent with the duality, i.e. the strict topology $\beta_0(E)$. Since this topology will also be of independent interest later on, we explain this in more detail. For a Polish space E , the *strict topology* $\beta_0(E)$ is defined as

follows. Let $\mathcal{F}_0(E)$ be the space of all bounded functions $\varphi : E \rightarrow \mathbb{C}$ which vanish at infinity, i.e. given $\varepsilon > 0$ there exists a compact set $K \subset E$ such that $|\varphi(x)| \leq \varepsilon$ for all $x \notin K$. The topology $\beta_0(E)$ is defined by the seminorms $(p_\varphi)_{\varphi \in \mathcal{F}_0}$, where $p_\varphi(f) = \|\varphi f\|_\infty$. Obviously, it suffices to consider only real-valued, positive φ .

It is well known that the strict topology coincides with the topology of uniform convergence on compact sets on $\|\cdot\|_\infty$ -bounded sets. Since transition semigroups are bounded, it follows that given a transition semigroup \mathbf{T} with the Feller property, for $f \in C_b(E)$ the orbit $t \mapsto T(t)f$ is continuous with respect to the compact-open topology if and only if it is continuous with respect to the topology $\beta_0(E)$. Furthermore, $\beta_0(E)$ is the strict Mackey topology of the dual pair $(C_b(E), \mathcal{M}(E))$, i.e. $\beta_0(E)$ is the topology of uniform convergence on the $\sigma(\mathcal{M}(E), C_b(E))$ -compact subsets of $\mathcal{M}(E)$. We refer to [40] (see also [24] for further references) for more information.

We will say that a transition semigroup \mathbf{T} is a *Feller semigroup* if it has the Feller property and the orbits of bounded, continuous functions under \mathbf{T} are pointwise continuous. We will say that \mathbf{T} is a *strictly continuous Feller semigroup* if \mathbf{T} has the Feller property and the orbits of bounded, continuous functions under \mathbf{T} are $\beta_0(E)$ -continuous. The importance of strictly continuous Feller semigroups arises from the fact that such semigroups are not only $\beta_0(E)$ -continuous, but also locally $\beta_0(E)$ -equicontinuous, see [24, Theorem 4.4]. This makes it possible to prove generation and perturbation results for such semigroups. Since the transition semigroup determines the distribution of solutions to the local martingale problem uniquely, this can be used to prove uniqueness of solutions to the local martingale problem, cf. [12].

Theorem 2.5. *Let \mathcal{L} be an admissible operator such that the local martingale problem for \mathcal{L} is completely well-posed and denote the associated semigroup by \mathbf{T} .*

- (1) *\mathbf{T} is a Feller semigroup if and only if $\mathbf{P}_{x_n} \xrightarrow{\text{fd}} \mathbf{P}_x$ whenever $x_n \rightarrow x$. Here, $\xrightarrow{\text{fd}}$ denotes weak convergence of the finite-dimensional distributions.*
- (2) *If $x \mapsto \mathbf{P}_x$ is continuous, then \mathbf{T} is a strictly continuous Feller semigroup. Here, $\mathcal{P}(C([0, \infty); E))$ is endowed with the weak topology.*

Proof. (1) It is easy to see that \mathbf{T} is a Feller semigroup if and only if the one-dimensional marginals of \mathbf{P}_{x_n} converge to those of \mathbf{P}_x . The result now follows from [11, Lemma 4.8.1].

(2) If $x \mapsto \mathbf{P}_x$ is continuous, \mathbf{T} is a Feller semigroup by part (1). By the results of [24, 25], $\mathbf{T}|_{C_b(E)}$ is an integrable semigroup on $(C_b(E), \mathcal{M}(E))$ with $\sigma(C_b(E), \mathcal{M}(E))$ -densely defined generator.

By [24, Theorem 4.4], to prove the $\beta_0(E)$ -continuity, it suffices to produce, given $T, \varepsilon > 0$ and a compact set $K \subset E$ a compact set $L \subset E$ such that

$$p_t(x, E \setminus L) \leq \varepsilon \quad \forall x \in K, t \in [0, T].$$

However, if $K \subset E$ is compact, then, by continuity of the map $x \mapsto \mathbf{P}_x$, the set of measures $\{\mathbf{P}_x : x \in K\}$ is weakly compact and thus, by Prokhorov's theorem, tight. Hence, given $\varepsilon > 0$ we find a compact set $\mathcal{C} \subset C([0, \infty); E)$ such that $\mathbf{P}_x(C([0, \infty); E) \setminus \mathcal{C}) \leq \varepsilon$ for all $x \in K$. However, by the Arzelà-Ascoli theorem, given $T > 0$ there exists a compact set $L \subset E$ such that $\mathbf{x}(t) \in L$ for all $\mathbf{x} \in \mathcal{C}$ and $0 \leq t \leq T$. Hence

$$p_t(x, E \setminus L) = \mathbf{P}_x(\mathbf{x}(t) \in E \setminus L) \leq \mathbf{P}_x(C([0, \infty); E) \setminus \mathcal{C}) \leq \varepsilon$$

for all $x \in K$ and $0 \leq t \leq T$. This finishes the proof. \square

3. STOCHASTIC DIFFERENTIAL EQUATIONS ON BANACH SPACES

We now turn our attention to the stochastic evolution equation (1.1). In order to stress the dependence on the coefficients, we will also refer to equation (1.1) as equation $[A, F, G]$. The following are our standing hypotheses on the coefficients and will be assumed throughout Section 3.

Hypothesis 3.1. \tilde{E} is a separable Banach space and A generates a strongly continuous semigroup $\mathbf{S} := (S(t))_{t \geq 0}$ on \tilde{E} . H is a separable Hilbert space and W_H is an H -cylindrical Wiener process. E is a separable Banach space such that $D(A) \subset E \subset \tilde{E}$ with continuous and dense embeddings. Throughout, all Banach spaces are real. Furthermore,

- (1) $F : E \rightarrow \tilde{E}$ is strongly measurable and bounded on bounded subsets of E ;
- (2) $G : E \rightarrow \mathcal{L}(H, \tilde{E})$ is H -strongly measurable, i.e. $Gh : E \rightarrow \tilde{E}$ is strongly measurable for all $h \in H$, and G is bounded on bounded subsets of E .

Let us recall that an H -cylindrical Wiener process is a bounded linear operator W_H from $L^2(0, \infty; H)$ to $L^2(\Omega, \Sigma, \mathbb{P})$, where $(\Omega, \Sigma, \mathbb{P})$ is a probability space, such that $W_H(f)$ is a centered Gaussian random variable for all $f \in L^2(0, \infty; H)$ and we have

$$\mathbb{E}(W_H(f_1)W_H(f_2)) = [f_1, f_2]_{L^2(0, \infty; H)}, \quad f_1, f_2 \in L^2(0, \infty; H).$$

We shall write

$$W_H(t)h := W_H(\mathbb{1}_{(0, t]} \otimes h), \quad t > 0, h \in H.$$

Note that for all $h \in H$ the process $W_H h := (W_H(t)h)_{t \geq 0}$ is a real-valued Brownian motion (which is standard if $\|h\|_H = 1$).

If a filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is specified on $(\Omega, \Sigma, \mathbb{P})$, we will say that W_H is an H -cylindrical Wiener process *with respect to* \mathbb{F} if $W_H h$ is a Brownian motion with respect to \mathbb{F} for all $h \in H$, i.e. the process $t \mapsto W_H h$ is \mathbb{F} -adapted and the increments $W_H(t)h - W_H(s)h$ are independent of \mathcal{F}_s .

We will use the following solution concept.

Definition 3.2. A tuple $((\Omega, \Sigma, \mathbb{P}), \mathbb{F}, W_H, \mathbf{X})$, where $(\Omega, \Sigma, \mathbb{P})$ is a probability space endowed with a filtration \mathbb{F} , W_H is an H -cylindrical Wiener process with respect to \mathbb{F} and \mathbf{X} is a continuous, \mathbb{F} -progressive, E -valued process is called *weak solution* of (1.1) if for all $x^* \in D(A^*) \subset \tilde{E}^*$ and $t \geq 0$ we have

$$\begin{aligned} \langle X(t), x^* \rangle &= \langle X(0), x^* \rangle + \int_0^t \langle X(s), A^* x^* \rangle ds \\ (3.1) \quad &+ \int_0^t \langle F(X(s)), x^* \rangle ds + \int_0^t G(X(s))^* x^* dW_H(s), \end{aligned}$$

\mathbb{P} -a.e.

Remark 3.3. Let us note that ‘solutions’ of (1.1) are *always* required to have continuous paths. This applies to weak solutions, as well as to other solution concepts, introduced later.

As a second remark, we note that weak solutions are weak *both* in the analytic (i.e. we require (1.1) to hold only if tested against functionals $x^* \in D(A^*)$) and in the stochastic sense, i.e. the probability space is part of the solution. In fact, all solution concepts that we will use will be weak in the stochastic sense. Hence, in order to shorten notation, we have decided to use the term ‘weak solution’ rather than ‘analytically weak and stochastically weak solution’ or ‘weak martingale solution’. Also this remark applies to other solution concepts that we will use.

By the continuity of the paths and our assumptions, the Lebesgue-integral in (3.1) is well defined. The stochastic integral in equation (3.1) is an integral of an $H \simeq H^*$ -valued stochastic processes with respect to a cylindrical Wiener process. It

is well known how to construct such an integral for adapted H -valued processes Φ such that $\Phi \in L^2(0, T; H)$ almost surely for all $T > 0$. Namely, if (h_k) is a (finite or countably infinite) orthonormal basis of H and we define $\beta_k(s) := W_H(s)h_k$, then

$$\int_0^t \Phi(s) dW_H(s) := \sum_k [\Phi(s), h_k]_H d\beta_k(s).$$

The integral process $\mathbf{I}(t) := \int_0^t \Phi(s) dW_H(s)$ is a continuous, local martingale with quadratic variation $[\mathbf{I}]_t = \int_0^t \|\Phi(s)\|_H^2 ds$. We also note that for a \mathbb{F} -stopping time τ we have almost surely $\mathbf{I}(t \wedge \tau) = \int_0^t \mathbb{1}_{[0, \tau]}(s) \Phi(s) dW_H(s)$ for all $t \geq 0$.

In order to shorten notation, we will say that a process \mathbf{X} is a weak solution of (1.1), meaning that \mathbf{X} is a continuous, progressive, E -valued process, defined on a stochastic basis $(\Omega, \Sigma, \mathbb{P}, \mathbb{F})$ on which an H -cylindrical Wiener process W_H with respect to \mathbb{F} exists such that $((\Omega, \Sigma, \mathbb{P}), \mathbb{F}, W_H, \mathbf{X})$ is a weak solution of (1.1). In this case, unless stated otherwise, \mathbb{P} will denote the measure on the probability space and W_H the H -cylindrical Wiener process. These remarks apply, mutatis mutandis, also for the other solution concepts that we will introduce.

Remark 3.4. We note that the exceptional set in (3.1) which initially depends on x^* and t may be chosen independently of t , since the deterministic integrals as well as the stochastic integral in (3.1) are pathwise continuous in t .

3.1. The strong Markov property. We now establish the strong Markov property for weak solutions of (1.1) using the results of Section 2. The complete, separable metric space we work on will be the Banach space E from Hypothesis 3.1 endowed with the metric induced by its norm. The admissible operator \mathcal{L} we will consider is defined in terms of the coefficients A, F and G . We then prove a one-to-one correspondence between solutions of the local martingale problem for this operator $\mathcal{L} = \mathcal{L}_{[A, F, G]}$ and weak solutions of (1.1). Hence, results about solutions of the local martingale problem for this operator \mathcal{L} yield corresponding results for weak solutions of equation $[A, F, G]$.

The operator $\mathcal{L}_{[A, F, G]}$ is defined as follows.

Let \mathcal{D} denote the vector space of all functions $f : E \rightarrow \mathbb{R}$ of the form

$$f(x) = \varphi(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle)$$

where $n \in \mathbb{N}$, $\varphi \in C^2(\mathbb{R}^n)$ and $x_1^*, \dots, x_n^* \in D(A^*)$.

For $f = \varphi(\langle \cdot, x_1^* \rangle, \dots, \langle \cdot, x_n^* \rangle) \in \mathcal{D}$ we put

$$(3.2) \quad \begin{aligned} L_{[A, F, G]} f(x) &:= \sum_{k=1}^n \frac{\partial \varphi}{\partial u_k}(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle) \cdot [\langle x, A^* x_k^* \rangle + \langle F(x), x_k^* \rangle] \\ &\quad + \frac{1}{2} \sum_{k, l=1}^n [G(x)^* x_k^*, G(x)^* x_l^*]_H \frac{\partial^2 \varphi}{\partial u_k \partial u_l}(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle) \end{aligned}$$

The operator $\mathcal{L}_{[A, F, G]}$ is defined by $D(\mathcal{L}) = \mathcal{D}$ and $\mathcal{L}_{[A, F, G]} f := L_{[A, F, G]} f$. Put $\mathcal{D}_{\min} := \{\langle \cdot, x^* \rangle^j : x^* \in D(A^*), j = 1, 2\}$. We will also use the operator $\mathcal{L}_{[A, F, G]}^{\min} := \mathcal{L}_{[A, F, G]}|_{\mathcal{D}_{\min}}$. We note that since F and G are bounded on bounded subsets of E , the operators $\mathcal{L}_{[A, F, G]}$ and $\mathcal{L}_{[A, F, G]}^{\min}$ are admissible.

Theorem 3.5. *Suppose that \mathbf{X} is a weak solution of equation $[A, F, G]$. Then the law \mathbf{P} of \mathbf{X} solves the local martingale problem for $\mathcal{L}_{[A, F, G]}$.*

Conversely, if \mathbf{P} solves the local martingale problem for $\mathcal{L}_{[A, F, G]}^{\min}$, then there exists a weak solution \mathbf{X} of equation $[A, F, G]$ with distribution \mathbf{P} .

Proof. First suppose that \mathbf{X} is a weak solution of equation $[A, F, G]$.

Let $f = \varphi(\langle \cdot, x_1^* \rangle, \dots, \langle \cdot, x_n^* \rangle) \in \mathcal{D}$ and define the \mathbb{R}^n -valued process ξ by

$$\xi_k(t) := \langle X(0), x_k^* \rangle + \int_0^t \langle X(s), A^* x_k^* \rangle + \langle F(X(s)), x_k^* \rangle ds + \int_0^t G(X(s))^* x_k^* dW_H(s),$$

for $k = 1, \dots, n$. Since X is a weak solution, for all $t \geq 0$ we have $\xi_k(t) = \langle X(t), x_k^* \rangle$ almost surely. Since ξ_k and $\langle X(\cdot), x_k^* \rangle$ are almost surely continuous, we actually may assume that almost surely $\xi_k(t) = \langle X(t), x_k^* \rangle$ for all $t \geq 0$ and $k = 1, \dots, n$.

We also define \mathbb{R}^n -valued processes V and M by

$$V_k(t) := \int_0^t \langle X(s), A^* x_k^* \rangle + \langle F(X(s)), x_k^* \rangle ds, \quad M_k(t) := \int_0^t G(X(s))^* x_k^* dW_H(s),$$

for $k = 1, \dots, n$. Note that, almost surely, V has continuous trajectories of locally bounded variation and that M is a continuous, local martingale. It follows from Itô's formula [11, Theorem 5.2.9] that, almost surely,

$$\begin{aligned} f(X(t)) - f(X(0)) &= \varphi(\xi(t)) - \varphi(\xi(0)) \\ &= \sum_{k=1}^n \int_0^t \frac{\partial \varphi}{\partial u_k}(\xi(s)) dV_k(s) + \sum_{k=1}^n \int_0^t \frac{\partial \varphi}{\partial u_k}(\xi(s)) dM_k(s) \\ &\quad + \frac{1}{2} \sum_{k,l=1}^n \int_0^t \frac{\partial^2 \varphi}{\partial u_k \partial u_l}(\xi(s)) d\llbracket M_k, M_l \rrbracket_s \\ &= \int_0^t [L_{[A,F,G]} f](X(s)) ds + \sum_{k=1}^n \int_0^t \frac{\partial \varphi}{\partial u_k}(\xi(s)) dM_k(s), \end{aligned}$$

for all $t \geq 0$. Here, we have used that

$$\llbracket M_k, M_l \rrbracket_t = \int_0^t [G(X(s))^* x_k^*, G(X(s))^* x_l^*]_H ds.$$

It thus follows that

$$f(X(t)) - f(X(0)) - \int_0^t [L_{[A,F,G]} f](X(s)) ds$$

is a continuous local martingale with respect to \mathbb{F} . Passing to the range space $C([0, \infty); E)$, it follows that under the distribution \mathbf{P} of \mathbf{X} , the process \mathbf{M}^f is a continuous local martingale.

We now prove the converse. First note that if $x^* \in D(A^*)$, then for $f_1(x) = \langle x, x^* \rangle$ we have $L_{[A,F,G]} f_1(x) = \langle x, A^* x^* \rangle + \langle F(x), x^* \rangle$ and for $f_2(x) = \langle x, x^* \rangle^2$ we have $L_{[A,F,G]} f_2(x) = 2\langle x, A^* x^* \rangle \cdot [\langle x, A^* x^* \rangle + \langle F(x), x^* \rangle] + \|G(x)^* x^*\|_H^2$. Since \mathbf{M}^{f_1} and \mathbf{M}^{f_2} are local martingales, it follows from [35, Lemma 34] that under \mathbf{P} the process

$$\langle \mathbf{x}(t), x^* \rangle - \langle \mathbf{x}(0), x^* \rangle - \int_0^t \langle \mathbf{x}(s), A^* x^* \rangle + \langle F(\mathbf{x}(s)), x^* \rangle ds$$

is a continuous local martingale with quadratic variation $\int_0^t \|G(\mathbf{x}(s))^* x^*\|_H^2 ds$. By [36, Theorem 3.1], we find an extension $(\Omega, \Sigma, \mathbb{F}, \mathbb{P})$ of $(C([0, \infty); E), \mathcal{B}, \mathbb{B}, \mathbf{P})$ on which a cylindrical Brownian motion W_H is defined such that for all $x^* \in D(A^*)$ we have

$$\langle \mathbf{x}(t), x^* \rangle - \langle \mathbf{x}(0), x^* \rangle - \int_0^t \langle \mathbf{x}(s), A^* x^* \rangle + \langle F(\mathbf{x}(s)), x^* \rangle ds = \int_0^t G(\mathbf{x}(s))^* x^* dW_H(s)$$

\mathbf{P} -almost everywhere for all $t \geq 0$. This proves that \mathbf{x} , defined on this extension, is a weak solution of $[A, F, G]$. \square

Corollary 3.6. *A measure $\mathbf{P} \in \mathcal{P}(C([0, \infty); E))$ solves the local martingale problem for $\mathcal{L}_{[A,F,G]}$ if and only if it solves the local martingale problem for $\mathcal{L}_{[A,F,G]}^{\min}$.*

Motivated by Theorem 3.5 we will say that the local martingale problem for $\mathcal{L}_{[A,F,G]}$ is the local martingale problem *associated with* equation $[A, F, G]$. We will say that equation $[A, F, G]$ is (completely) well-posed if the associated local martingale problem is (completely) well-posed.

We next prove that if equation $[A, F, G]$ is well-posed, then it is completely well-posed. By the results of the previous section, this implies that there exists a transition semigroup \mathbf{T} such that every solution of $[A, F, G]$ is a strong Markov process with transition semigroup \mathbf{T} .

As in the finite dimensional case, cf. [11, Theorem 4.4.6], the key step is to prove that it even suffices to consider the martingale problem for an operator $\mathcal{L}_{[A,F,G]}^0$, defined on a countable set.

Lemma 3.7. *There exists a countable subset \mathcal{D}_0 of \mathcal{D} such that a measure \mathbf{P} solves the local martingale problem associated for $\mathcal{L}_{[A,F,G]}$ if and only if it solves the martingale problem associated with $\mathcal{L}_{[A,F,G]}^0 := \mathcal{L}_{[A,F,G]}|_{\mathcal{D}_0}$.*

Proof. Step 1: We construct the set \mathcal{D}_0 .

First note that there exists a countable subset D of $D(A^*)$ such that for every $x^* \in D(A^*)$ there exists a sequence $(x_n^*) \subset D$ such that $x_n^* \rightharpoonup^* x^*$ and $A^*x_n^* \rightharpoonup^* A^*x^*$. To see this, first note that there is a countable set $\{z_n^* : n \in \mathbb{N}\} \subset \tilde{E}^*$ which is sequentially weak*-dense in \tilde{E}^* , see §21.3 (5) of [21]. Put $D := \{R(\lambda, A^*)z_n^* : n \in \mathbb{N}\}$ for some $\lambda \in \rho(A^*)$. Using that $R(\lambda, A^*)$ is $\sigma(\tilde{E}^*, \tilde{E})$ -continuous as an adjoint operator, it is easy to see that D has the required properties. Replacing D with the set of all convex combinations of elements of D with rational coefficients, we may (and shall) assume that such convex combinations belong to D again.

Now choose a sequence $\varphi_n \in C^2(\mathbb{R})$ with the following properties:

- (1) $\varphi_n(t) = t$ for all $-n \leq t \leq n$ and $\varphi_n(t) = 0$ for $t \notin [-2n, 2n]$.
- (2) $\sup_n \|\varphi_n'\|_\infty, \sup_n \|\varphi_n''\|_\infty < \infty$.

We then define

$$\mathcal{D}_0 := \{f = \varphi_n(\langle \cdot, x^* \rangle)^j \mid \text{for some } n \in \mathbb{N}, x^* \in D, j \in \{1, 2\}\}$$

and put $\mathcal{L}_{[A,F,G]}^0 := \mathcal{L}_{[A,F,G]}|_{\mathcal{D}_0}$.

Step 2: Now let \mathbf{P} be a solution of the local martingale problem for $\mathcal{L}_{[A,F,G]}^0$. We prove that \mathbf{P} solves the local martingale problem for $\mathcal{L}_{[A,F,G]}^{\min}$. This finishes the proof in view of Corollary 3.6.

First note that \mathbf{M}^f is a local martingale for any $f = \langle \cdot, x^* \rangle^j$, $x^* \in D$, $j \in \{1, 2\}$. To see this, let $\tau_n := \inf\{t > 0 : |\langle \mathbf{x}(t), x^* \rangle| \geq n\}$ and put $f_n := \varphi_n(\langle \cdot, x^* \rangle)^j \in \mathcal{D}_0$. Arguing as in Lemma 2.2, it follows that $\mathbf{M}_{\tau_n}^{f_n} = \mathbf{M}_{\tau_n}^f$ is a martingale under \mathbf{P} .

Now fix $x^* \in D(A^*)$ and a sequence $(x_n^*) \subset D$ such that $x_n^* \rightharpoonup^* x^*$ and $A^*x_n^* \rightharpoonup^* A^*x^*$. By the uniform boundedness principle, the sequences (x_n^*) and $(A^*x_n^*)$ are bounded in \tilde{E}^* , say by M . For $m \in \mathbb{N}$ put $\tau_m := \inf\{t > 0 : \|\mathbf{x}(t)\| \geq m\}$.

Let us first consider $f := \langle \cdot, x^* \rangle$. For $f_n := \langle \cdot, x_n^* \rangle$, the stopped process $\mathbf{M}_{\tau_m}^{f_n}$ is a martingale for all $n, m \in \mathbb{N}$. Furthermore, since $L_{[A,F,G]}f_n \rightarrow L_{[A,F,G]}f$ pointwise, it follows that $\mathbf{M}_{\tau_m}^{f_n}(t) \rightarrow \mathbf{M}_{\tau_m}^f(t)$ pointwise, for all $t \geq 0$. Since F is bounded on $\bar{B}(0, m)$, say by C_m , we find for $t > s$

$$|\mathbf{M}_{\tau_m}^{f_n}(\mathbf{x})(t) - \mathbf{M}_{\tau_m}^{f_n}(\mathbf{x})(s)| \leq (t-s)[m \cdot M + C_m \cdot M] + 2m \cdot M$$

for all $n, m \in \mathbb{N}$. Applying the dominated convergence theorem to $(\mathbf{M}_{\tau_m}^{f_n}(t) - \mathbf{M}_{\tau_m}^{f_n}(s))\mathbb{1}_B$, where B is an arbitrary set in \mathcal{B}_s , it follows that $\mathbf{M}_{\tau_m}^f$ is a \mathbb{B} -martingale under \mathbf{P} . Since $\tau_m \uparrow \infty$ almost surely, \mathbf{M}^f is a local martingale.

Next consider $f := \langle \cdot, x^* \rangle^2$. For $f_n := \langle \cdot, x_n^* \rangle^2$, the stopped process $\mathbf{M}_{\tau_m}^{f_n}$ is a martingale for all $n, m \in \mathbb{N}$. Similarly as above, one sees that for every $m \in \mathbb{N}$ the difference $|\mathbf{M}_{\tau_m}^{f_n}(t) - \mathbf{M}_{\tau_m}^{f_n}(s)|$ may be majorized by a bounded function independent

of n . However, due to the term $\|G(\cdot)^* x_n^*\|_H^2$ in $L_{[A,F,G]} f_n$, the weak convergence $x_n^* \rightharpoonup^* x^*$ does not suffice to conclude that $L_{[A,F,G]} f_n \rightarrow L_{[A,F,G]} f$ pointwise. Hence we employ a different method here. A similar argument was used in the proof of [33, Theorem 2.3], cf. also [36, Corollary 2.3].

We fix $0 \leq s < t$ and $m \in \mathbb{N}$. The dominated convergence theorem yields weak convergence

$$\int_s^t \mathbb{1}_{[0,\tau_m]}(\mathbf{x}(r)) G(\mathbf{x}(s))^* x_n^* dr \rightharpoonup \int_s^t \mathbb{1}_{[0,\tau_m]}(\mathbf{x}(r)) G(\mathbf{x}(s))^* x^* dr \quad \text{in } L^2(\Omega, \mathbf{P}; H) .$$

Hence, for any $N \in \mathbb{N}$, $\int_s^t \mathbb{1}_{[0,\tau_m]}(\mathbf{x}(r)) G(\mathbf{x}(r))^* x^* dr$ belongs to the weak closure of the tail sequence $(\int_s^t \mathbb{1}_{[0,\tau_m]}(\mathbf{x}(r)) G(\mathbf{x}(r))^* x_n^* dr)_{n \geq N}$. By the Hahn-Banach theorem, it belongs to the strong closure of that tail, whence we find vectors y_N^* , belonging to the convex hull the sequence $(x_n^*)_{n \geq N}$, such that we have *strong* convergence

$$\int_s^t \mathbb{1}_{[0,\tau_m]}(\mathbf{x}(r)) G(\mathbf{x}(r))^* y_N^* dr \rightarrow \int_s^t \mathbb{1}_{[0,\tau_m]}(\mathbf{x}(r)) G(\mathbf{x}(r))^* x^* dr \quad \text{in } L^2(\Omega, \mathbf{P}; H) .$$

After passing to a subsequence, we may assume that this convergence holds pointwise a.e. Note that $y_N^* \rightharpoonup^* x^*$, as y_N^* belongs to the tail $(x_n^*)_{n \geq N}$. Hence it follows that

$$\mathbf{M}_{\tau_m}^{g_N}(t) - \mathbf{M}_{\tau_m}^{g_N}(s) \rightarrow \mathbf{M}_{\tau_m}^f(t) - \mathbf{M}_{\tau_m}^f(s)$$

pointwise almost everywhere. Here, $g_N := \langle \cdot, y_N^* \rangle^2$.

Note that we may assume without loss of generality that y_N^* is a convex combination of the $(x_n^*)_{n \geq N}$ with rational coefficients. Hence, $y_N^* \in D$ and thus $g_N \in \mathcal{D}_0$, implying that $\mathbf{M}_{\tau_m}^{g_N}$ is a martingale for all $N \in \mathbb{N}$. Now, similarly as above, the dominated convergence theorem shows that $\mathbf{M}_{\tau_m}^f$ is a martingale for all $m \in \mathbb{N}$. This finishes the proof. \square

Now the announced result about the equivalence of well-posedness and complete well-posedness follows as in the finite-dimensional case, see [19, Theorem 21.10]. For reasons of completeness, we include a proof.

Theorem 3.8. *Suppose that the local martingale problem for $\mathcal{L}_{[A,F,G]}$ is well-posed. Then it is completely well-posed. Consequently, all weak solutions of equation $[A, F, G]$ are strong Markov processes with a common transition semigroup \mathbf{T} .*

Proof. We first prove the measurability of the map $x \mapsto \mathbf{P}_x$. Consider the set $V := \{\mathbf{P}_x : x \in E\}$. We claim that V is a Borel subset of $\mathcal{P}(C([0, \infty); E))$. Indeed, by well-posedness, $V = V_1 \cap V_2$, where V_1 is the set of all probability measures with degenerate initial distributions and V_2 is the set of all solutions to the martingale problem.

Since the map $\mathbf{P} \mapsto \mathbf{P} \circ \mathbf{x}(0)^{-1}$ is measurable from $\mathcal{P}(C([0, \infty); E))$ to $\mathcal{P}(E)$, the measurability of V_1 follows from [19, Lemma 1.39].

By Lemma 3.7, $\mathbf{P} \in V_2$ if and only if \mathbf{M}^f is a local martingale for all $f \in \mathcal{D}_0$. With $\tau_n := \inf\{t > 0 : \|\mathbf{x}(t)\| \geq n\}$, this is equivalent with

$$\int_B \mathbf{M}^f(t \wedge \tau_n) d\mathbf{P} = \int_B \mathbf{M}^f(s \wedge \tau_n) d\mathbf{P} \quad \forall s < t, B \in \mathcal{B}_s, n \in \mathbb{N} .$$

However, using continuity of $t \mapsto \mathbf{x}(t)$ and the fact that the σ -algebra \mathcal{B}_s is countably generated for all $s > 0$, we see that \mathbf{M}^f is a local martingale whenever the above equality holds for $n \in \mathbb{N}, s, t \in \mathbb{Q}$ with $s < t$ and B in a countable subset of \mathcal{B}_s . Hence the set V_2 is determined by countably many ‘measurable relations’ and hence measurable. It follows that V is measurable as claimed.

Now define the map $\Phi : V \rightarrow E$ by defining $\Phi(\mathbf{P})$ as the unique x such that $\mathbf{P} \circ \mathbf{x}(0)^{-1} = \delta_x$. Clearly, Φ is injective. Furthermore, Φ is measurable as the

composition of the measurable map $\mathbf{P} \circ \mathbf{x}(0)^{-1}$ and the inverse of the map $x \mapsto \delta_x$, which establishes a homeomorphism between E and the range of that map. By the Kuratowski Theorem, see [38, Section 1.3], the inverse Φ^{-1} is measurable, i.e. $x \mapsto \mathbf{P}_x$ is a measurable map from E to $\mathcal{P}(C([0, \infty); E))$

It remains to prove the uniqueness of solutions with arbitrary initial distributions μ for the martingale problem for $\mathcal{L}_{[A, F, G]}$, the existence of solutions with general initial distributions will then follow from Theorem 2.3.

To that end, assume that \mathbf{P} solves the local martingale problem for $\mathcal{L}_{[A, F, G]}$ and that $\mathbf{x}(0)$ has distribution $\mu \in \mathcal{P}(E)$. Let $\mathbf{Q} : E \times \mathcal{B} \rightarrow [0, 1]$ be a regular conditional probability (under \mathbf{P}) for \mathcal{B} given $\mathbf{x}(0)$. Then

$$\mathbf{P}(A) = \int_E \mathbf{Q}(x, A) d\mu(x) \quad \forall A \in \mathcal{B}.$$

Now let $t > s \geq 0$ and $B \in \mathcal{B}_s$ be given. Then, for $f \in \mathcal{D}$, we have

$$\int_B \mathbf{M}^f(t \wedge \tau_n) - \mathbf{M}^f(s \wedge \tau_n) d\mathbf{Q}(x, \cdot) = \int_{B \cap \{\mathbf{x}(0)=x\}} \mathbf{M}^f(t \wedge \tau_n) - \mathbf{M}^f(s \wedge \tau_n) d\mathbf{P} = 0$$

\mathbf{P} -almost everywhere. We note that the null-set outside of which this equation holds depends on t, s, n, B and the function f . However, arguing as above, we see that for fixed f , there exists a null-set $N(f)$, such that the above equation holds outside $N(f)$ for all $t > s, n \in \mathbb{N}$ and $B \in \mathcal{B}_s$. Putting $N := \bigcup_{f \in \mathcal{D}_0} N(f)$, it follows that outside of N , the above holds for all $t > s, n \in \mathbb{N}, B \in \mathcal{B}_s$ and $f \in \mathcal{D}_0$. This implies that for μ -a.e. x the measure $\mathbf{Q}(x, \cdot)$ solves the local martingale problem for $\mathcal{L}_{[A, F, G]}^0$ and hence, by Lemma 3.7, the local martingale problem for $\mathcal{L}_{[A, F, G]}$. By well-posedness, $\mathbf{Q}(x, \cdot) = \mathbf{P}_x(\cdot)$ for μ -a.e. x . Hence we have

$$\mathbf{P}(A) = \int_E \mathbf{P}_x d\mu(x) \quad \forall A \in \mathcal{B},$$

This shows that uniqueness of solutions of the local martingale problem for (\mathcal{L}, δ_x) for all $x \in E$ implies uniqueness of the solution of the local martingale problem for (\mathcal{L}, μ) for arbitrary initial distribution μ . \square

Next we give a lemma which allows us to construct solutions to $[A, F, G]$. This lemma is also helpful in verifying the continuity of the map $x \mapsto \mathbf{P}_x$ and thus to prove the Feller property and the strict continuity of the associated transition semigroup.

Lemma 3.9. *Suppose we are given sequences $(F_n)_{n \in \mathbb{N}}$ and $(G_n)_{n \in \mathbb{N}}$ which satisfy the assumptions of Hypothesis 3.1 and are uniformly bounded on bounded sets. Furthermore, assume that F_n converges to F and, for every $h \in H$, $G_n h$ converges to Gh , both convergences being uniform on the compact subsets of E . If \mathbf{P}_n solves the martingale problem associated with equation $[A, F_n, G_n]$ and if the sequence $(\mathbf{P}_n)_{n \in \mathbb{N}}$ is tight, then any accumulation point of the sequence solves the martingale problem associated with $[A, F, G]$.*

Proof. For $M \in \mathbb{N}$ we put $\tau_M := \inf\{t > 0 : \|\mathbf{x}(t)\| \geq M\}$. Now fix $0 \leq s_1 < \dots < s_N \leq s < t$, $M \in \mathbb{N}$, for $j = 1, \dots, N$, functions $h_j \in C_b(E)$ and $f = \varphi(\langle \cdot, x_1^* \rangle, \dots, \langle \cdot, x_m^* \rangle) \in \mathcal{D}$.

We define $\Phi_n : C([0, \infty); E) \rightarrow \mathbb{R}$ by

$$\Phi_n(\mathbf{x}) := \left[f(\mathbf{x}(t \wedge \tau_M)) - f(\mathbf{x}(s \wedge \tau_M)) - \int_s^t \mathbb{1}_{[0, \tau_M]}(r) (L_n f)(\mathbf{x}(r)) dr \right] \cdot \prod_{j=1}^N h_j(\mathbf{x}(s_j)),$$

where $L_n := L_{[A, F_n, G_n]}$. Similarly, we define the function Φ , replacing L_n with $L := L_{[A, F, G]}$.

Clearly, Φ_n is a continuous function for every $n \in \mathbb{N}$. Using the assumption that F_n and G_n are uniformly bounded on bounded subsets, it is easy to see that the sequence Φ_n is uniformly bounded.

The assumptions on the convergence of F_n and G_n imply that $L_n f$ converges to Lf , uniformly on the compact subsets of E . Now let a compact subset \mathcal{C} of $C([0, \infty); E)$ be given. By the Arzelà-Ascoli theorem, there exists a compact subset K of E such that $\mathbf{x}(r) \in K$ for all $0 \leq r \leq t$, whenever $\mathbf{x} \in \mathcal{C}$. Let $C := \prod_{j=1}^n \|h_j\|_\infty$. Given $\varepsilon > 0$, pick n_0 such that $|L_n f(x) - Lf(x)| \leq \varepsilon$ for all $x \in K$, whenever $n \geq n_0$. Then, for $\mathbf{x} \in \mathcal{C}$ and $n \geq n_0$ we have

$$|\Phi_n(\mathbf{x}) - \Phi(\mathbf{x})| \leq \int_s^t \mathbb{1}_{[0, \tau_M]}(r) |L_n f(\mathbf{x}(r)) - Lf(\mathbf{x}(r))| dr \cdot C \leq |t - s| \varepsilon C,$$

proving that Φ_n converges to Φ uniformly on compact subsets of $C([0, \infty); E)$. In view of the uniform boundedness of the sequence Φ_n , it follows that $\Phi_n \rightarrow \Phi$ with respect to $\beta_0(C([0, \infty); E))$.

Now let \mathbf{P} be an accumulation point of the sequence (\mathbf{P}_n) . Passing to a subsequence, we may assume that \mathbf{P}_n converges weakly to \mathbf{P} . Since the sequence (\mathbf{P}_n) is tight, p defined by $p(\Psi) := \sup_n \left| \int \Psi d\mathbf{P}_n \right|$ is a $\beta_0(C([0, \infty); E))$ -continuous seminorm. Hence

$$\left| \int \Phi d\mathbf{P} - \int \Phi_n d\mathbf{P}_n \right| \leq \left| \int \Phi d\mathbf{P} - \int \Phi d\mathbf{P}_n \right| + p(\Phi - \Phi_n) \rightarrow 0$$

as $n \rightarrow \infty$. Thus, $\int \Phi d\mathbf{P} = \lim_{n \rightarrow \infty} \int \Phi_n d\mathbf{P}_n = 0$, since \mathbf{P}_n solves the martingale problem for $\mathcal{L}_{[A, F_n, G_n]}$. Since the sampling points (s_j) and s, t as well as the functions h_j were arbitrary, it follows from a monotone class argument that

$$f(\mathbf{x}(t \wedge \tau_M)) - \int_0^t \mathbb{1}_{[0, \tau_M]}(r) Lf(\mathbf{x}(r)) dr$$

is a martingale under \mathbf{P} . Since M and f were arbitrary, \mathbf{P} solves the martingale problem associated with equation $[A, F, G]$. \square

3.2. Pathwise uniqueness. In view of Theorem 3.5, the uniqueness requirement for the local martingale problem associated with (1.1) is equivalent with the requirement that any two weak solutions \mathbf{X}_1 and \mathbf{X}_2 of (1.1) such that $X_1(0)$ and $X_2(0)$ have the same distribution μ induce the same measure on $C([0, \infty); E)$. In this case, we will say that *uniqueness in law* or *uniqueness in distribution* holds for solutions with initial distribution μ .

In some cases, a different uniqueness concept is more appropriate.

Definition 3.10. We say that *pathwise uniqueness* holds for solutions of equation (1.1) with initial distribution μ if whenever $((\Omega, \Sigma, \mathbb{P}), \mathbb{F}, W_H, \mathbf{X}_j)$ are weak solution of (1.1) for $j = 1, 2$ with $X_1(0) = X_2(0) \sim \mu$, then $\mathbb{P}(X_1(t) = X_2(t) : \forall t \geq 0) = 1$.

A classical result of Yamada and Watanabe [44] asserts that in the case where $E = \mathbb{R}^d$ and W_H is a finite dimensional Brownian motion, i.e. H is finite-dimensional, pathwise uniqueness implies uniqueness in law. Ondreját [34] has generalized this result to the situation where E is a 2-smoothable Banach space. This result also holds in our more general situation.

Theorem 3.11. *Pathwise uniqueness for (1.1) implies uniqueness in law.*

The proof follows the classical lines of [44]. Here we just give a rough sketch of the proof and leave the details to the reader.

Given two weak solutions with identical initial distribution, one uses regular conditional probabilities to define distributional copies of the solutions on a common stochastic basis (on which an H -cylindrical Wiener process is defined) and having

the same initial value. The main task in the Banach space case is to prove that distributional copies of ‘solutions’ are again ‘solutions’. This depends on results about the distribution of random integrals. In [34], mild solutions were considered. In our situation, we consider weak solutions, whence there are only stochastic integrals of H -valued processes involved in our solution concept. Thus the results of [34] are sufficient to conclude that distributional copies of weak solutions are again weak solutions. Pathwise uniqueness yields that the two solutions are equal almost surely and from this one infers that the original solutions have the same distribution.

3.3. Weakly mild solutions. While the notion of weak solutions of (1.1) is appropriate to discuss the associated martingale problem and thus to establish the strong Markov property of solutions, experience has taught that in order to *produce* solutions to (1.1), the concept of *mild solutions* is better suited. However, this concept requires an E -valued stochastic integral. As an intermediate step, we consider a ‘weak notion’ of mild solution and postpone the discussion of stochastic integrability to the next section.

Definition 3.12. A tuple $((\Omega, \Sigma, \mathbb{P}), \mathbb{F}, W_H, \mathbf{X})$, where $(\Omega, \Sigma, \mathbb{P})$ is a probability space endowed with a filtration \mathbb{F} , W_H is an H -cylindrical Wiener process with respect to \mathbb{F} and \mathbf{X} is a continuous, \mathbb{F} -progressive, E -valued process is called a *weakly mild solution* of (1.1) if for all $x^* \in E^*$ and $t \geq 0$ we have

$$(3.3) \quad \begin{aligned} \langle X(t), x^* \rangle &= \langle S(t)X(0), x^* \rangle + \int_0^t \langle S(t-s)F(X(s)), x^* \rangle ds \\ &\quad + \int_0^t G(X(s))^* S(t-s)^* x^* dW_H(s). \end{aligned}$$

\mathbb{P} -a.e.

Let us first prove that this is well-defined.

Lemma 3.13. *The Lebesgue-integral and the stochastic integral in (3.3) are well-defined for all $t \geq 0$ and $x^* \in E^*$.*

Proof. Let us first discuss the Lebesgue integral. The map $(s, \omega) \mapsto F(X(s, \omega))$ is measurable as a composition of two measurable maps. Hence, it is the limit of a sequence of simple functions f_n almost everywhere with respect to $ds \otimes \mathbb{P}$. Thus

$$\langle S(t - \cdot)F(X), x^* \rangle = \lim \langle f_n, S(t - \cdot)^* x^* \rangle \quad ds \otimes \mathbb{P} - a.e.$$

However, $\langle f_n, S(t - \cdot)^* x^* \rangle = \sum_{j=1}^{N_n} \mathbb{1}_{A_{jn}} \langle x_{jn}, S(t - \cdot)^* x^* \rangle$ for certain measurable sets A_{jn} and vectors $x_{jn} \in \tilde{E}$. This is measurable since $s \mapsto \langle x, S(t - s)^* x^* \rangle$ is continuous for all $x \in \tilde{E}$ and $x^* \in \tilde{E}^*$. Hence $\langle S(t - \cdot)F(X), x^* \rangle$ is the limit of measurable functions $ds \otimes \mathbb{P}$ almost everywhere and thus measurable. In view of the continuity of the paths of \mathbf{X} , the boundedness of F on bounded sets and the boundedness of \mathbf{S} on finite time intervals, it follows that for almost all ω the function $s \mapsto \langle S(t - s)F(X(s, \omega)), x^* \rangle$ is bounded, hence integrable.

For the stochastic integral we use the series expansion

$$G(X(s, \omega))^* S(t - s)^* x^* = \sum_k [G(X(s, \omega)h_k, S(t - s)^* x^*)]_H h_k$$

where (h_k) is a finite or countably infinite orthonormal basis of H . Similarly as above, it is seen that each summand in the series is adapted and, as a function of s , bounded almost surely. In fact, since $G : E \rightarrow \tilde{E}$ is bounded on bounded sets, the series converges almost surely in $L^2(0, t; H)$. This implies that $(s, \omega) \mapsto G(X(s, \omega))^* S(t - s)^* x^*$ is an adapted H -valued process which belongs to $L^2(0, t; H)$ almost surely. Consequently, the stochastic integral is well-defined. \square

We now prove that the notions ‘weak solution’ and ‘weakly mild solution’ are equivalent. Variations of this result (for mild solutions) have been proved in various settings, see [10, Theorem 5.4], [33, Theorem 7.1] or [39, Proposition 3.3]. In Section 4, we will present several geometric assumptions on E ensuring that a weakly mild solution is a mild solution.

We note that the adjoint semigroup \mathbf{S}^* may not be strongly continuous, which causes technical difficulties. To overcome these, we will use results about the \odot -dual semigroup \mathbf{S}^\odot . We recall some basic definitions and properties and refer the reader to [28] for more information.

Define $E^\odot := \overline{D(A^*)}$. Then E^\odot is a closed, weak*-dense subspace of \tilde{E}^* which is invariant under the adjoint semigroup. The restriction of the adjoint semigroup to E^\odot , denoted by \mathbf{S}^\odot , is strongly continuous. In fact, $E^\odot = \{x^* : t \mapsto S(t)^*x^* \text{ is strongly continuous}\}$. We denote by A^\odot the generator of \mathbf{S}^\odot . Note that A^\odot is exactly the part of A^* in E^\odot .

Proposition 3.14. *The weak and the weakly mild solutions of (1.1) coincide.*

Proof. First assume that \mathbf{X} is a weak solution. For $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t > 0 : \|X(t)\| \geq n\}.$$

Since \mathbf{X} is a weak solution, we have for $x^* \in D(A^*)$ and $t \geq 0$

$$\begin{aligned} \langle X(t \wedge \tau_n), x^* \rangle &= \langle X(0 \wedge \tau_n), x^* \rangle + \int_0^t \mathbb{1}_{[0, \tau_n]}(s) \langle X(s), A^* x^* \rangle ds \\ &\quad + \int_0^t \mathbb{1}_{[0, \tau_n]}(s) \langle F(X(s)), x^* \rangle ds + \int_0^t \mathbb{1}_{[0, \tau_n]}(s) G(X(s))^* x^* dW_H(s) \end{aligned}$$

almost surely. In view of Remark 3.4, we may (and shall) assume that the exceptional set does not depend on t . Below, we will suppress the statement \mathbb{P} -almost surely.

Fix $t > 0$ and let $f \in C^1([0, t])$ and $x^* \in D(A^*)$. Putting $\varphi := f \otimes x^*$, Itô’s formula yields

$$\begin{aligned} \langle X(t \wedge \tau_n), \varphi(t) \rangle &= \langle X(0 \wedge \tau_n), \varphi(0) \rangle + \int_0^t \mathbb{1}_{[0, \tau_n]}(s) \langle X(s), \varphi'(s) \rangle ds \\ (3.4) \quad &+ \int_0^t \mathbb{1}_{[0, \tau_n]}(s) \langle X(s), A^* \varphi(s) \rangle ds + \int_0^t \mathbb{1}_{[0, \tau_n]}(s) \langle F(X(s)), \varphi(s) \rangle ds \\ &+ \int_0^t \mathbb{1}_{[0, \tau_n]}(s) G(X(s))^* \varphi(s) dW_H(s). \end{aligned}$$

By linearity, the above equation holds for $\varphi = \sum_{k=1}^N f_k \otimes x_k^*$ where $f_k \in C^1([0, t])$ and $x_k^* \in D(A^*)$. Since $D(A^\odot)$ is a Banach space with respect to the graph norm, so is $C^1([0, t]; D(A^\odot))$. Functions of the form $\varphi := \sum_{k=1}^n f_k \otimes x_k^*$ with $f_k \in C^1([0, t])$ and $x_k^* \in D(A^\odot)$ for $1 \leq k \leq n$ are dense in $C^1([0, t]; D(A^\odot))$ and hence an approximation argument shows that (3.4) holds for all $\varphi \in C^1([0, t]; A^\odot)$.

Now let $x^* \in D((A^\odot)^2)$ and $\varphi(s) = S(t-s)^* x^*$. Then $\varphi \in C^1([0, t]; D(A^\odot))$ with $\varphi'(s) = -S(t-s)^* A^* x^*$. Thus (3.4) yields for this φ

$$\begin{aligned} \langle X(t \wedge \tau_n), x^* \rangle &= \langle S(t)X(0 \wedge \tau_n), x^* \rangle + \int_0^t \mathbb{1}_{[0, \tau_n]}(s) \langle S(t-s)F(X(s)), x^* \rangle ds \\ (3.5) \quad &+ \int_0^t \mathbb{1}_{[0, \tau_n]}(s) G(X(s))^* S(t-s)^* x^* dW_H(s). \end{aligned}$$

We now extend (3.5) to arbitrary $x^* \in \tilde{E}^*$. Since $D((A^\odot)^2)$ is sequentially weak*-dense in \tilde{E}^* , given $z^* \in \tilde{E}^*$, we find a sequence $x_k^* \in D((A^\odot)^2)$ such that $x_k^* \rightharpoonup^* z^*$.

Arguing similar as in the proof of Lemma 3.7, we find a sequence y_m^* in the convex hull of the (x_k^*) such that $y_m^* \rightharpoonup^* z^*$ and

$$\mathbb{1}_{[0, \tau_n]}(\cdot) G(X(\cdot))^* S(t - \cdot)^* y_m^* \rightarrow \mathbb{1}_{[0, \tau_n]}(\cdot) G(X(\cdot))^* S(t - \cdot)^* z^*$$

in $L^2(\Omega; L^2([0, t]; H))$. Thus, since $\mathbb{E} \left| \int_0^t \Phi(s) dW_H(s) \right|^2 = \|\Phi\|_{L^2(\Omega; L^2([0, t]; H))}^2$ we see that

$$\int_0^t \mathbb{1}_{[0, \tau_n]} G(X(s))^* S(t - s)^* y_m^* dW_H(s) \rightarrow \int_0^t \mathbb{1}_{[0, \tau_n]} G(X(s))^* S(t - s)^* z^* dW_H(s)$$

in $L^2(\Omega)$. Passing to a subsequence, we may assume that we have convergence almost everywhere. However, since (3.5) also holds for $x^* = y_m^*$, for all $m \in \mathbb{N}$, noting that

$$\mathbb{1}_{[0, \tau_n]}(s) |\langle S(t - s) F(X(s)), y_m^* \rangle| \leq \mathbb{1}_{[0, \tau_n]}(s) M e^{\omega(t-s)} B_n \cdot \sup_{m \in \mathbb{N}} \|y_m^*\|,$$

where M and ω are such that $\|S(t)\| \leq M e^{\omega t}$ for $t \geq 0$ and $B_n := \sup\{\|F(x)\| : \|x\| \leq n\}$, the dominated convergence theorem yields (3.5) for $x^* = z^*$. Upon letting $n \rightarrow \infty$, (3.3) is proved for arbitrary $x^* = z^*$.

We now prove the converse and assume that \mathbf{X} is a weakly mild solution of (3.1). Fix $x^* \in D(A^*)$ and $t > 0$. Then for $0 < s < t$ we have

$$(3.6) \quad \begin{aligned} \langle X(s), A^* x^* \rangle &= \langle S(s) X(0), A^* x^* \rangle + \int_0^s \langle S(s - r) F(X(r)), A^* x^* \rangle dr \\ &\quad + \int_0^s G(X(r))^* S(s - r)^* A^* x^* dW_H(r) \end{aligned}$$

almost surely. We note that the exceptional set may depend s . However, all terms in this equation are jointly measurable in s and ω . Hence, the left-hand side and the right-hand side of (3.6) are equal as elements of $L^0((0, t); L^0(\Omega))$. By the canonical isomorphism $L^0((0, t); L^0(\Omega)) \simeq L^0(\Omega; L^0(0, t))$, there exists a set $N \subset \Omega$ with $\mathbb{P}(N) = 0$ such that outside N equation (3.6) holds as an equation in $L^0(0, t)$, i.e. for almost every $s \in (0, t)$, where the exceptional set may depend on ω . Next note that by the continuity of the paths, the local boundedness of \mathbf{S} and the boundedness of F on bounded sets, the first three terms are, as functions of s , \mathbb{P} -almost surely bounded on $(0, t)$ and hence belong to $L^1(0, t)$. Possibly enlarging N , we may assume that outside N equation (3.6) holds as an equation in $L^1(0, t)$. Integrating from 0 to t , we find that, \mathbb{P} -almost surely, we have

$$(3.7) \quad \begin{aligned} \int_0^t \langle X(s), A^* x^* \rangle ds &= \int_0^t \langle S(s) X(0), A^* x^* \rangle ds \\ &\quad + \int_0^t \int_0^s \langle S(s - r) F(X(r)), A^* x^* \rangle dr ds \\ &\quad + \int_0^t \int_0^s G(X(r))^* S(s - r)^* A^* x^* dW_H(r) ds. \end{aligned}$$

Recall that for $x^* \in D(A^*)$ we have $\int_0^t S(s) A^* x^* ds = S(t) x^* - x^*$ for all $t \geq 0$. Here, the integral has to be understood as weak*-integral. Using this, we obtain, pathwise,

$$\begin{aligned} \int_0^t \langle S(s) X(0), A^* x^* \rangle ds &= \left\langle X(0), \int_0^t S(s)^* A^* x^* ds \right\rangle = \langle X(0), S(t)^* x^* - x^* \rangle \\ &= \langle S(t) X(0), x^* \rangle - \langle X(0), x^* \rangle. \end{aligned}$$

Using Fubini's theorem, we have

$$\begin{aligned} \int_0^t \int_0^s \langle S(s-r)F(X(r)), A^*x^* \rangle dr ds &= \int_0^t \left\langle F(X(r)), \int_r^t S(s-r)^* A^*x^* \right\rangle ds dr \\ &= \int_0^t \langle S(t-r)F(X(r)), x^* \rangle dr - \int_0^t \langle F(X(r)), x^* \rangle dr \end{aligned}$$

pathwise. Using the stochastic Fubini theorem [30, Theorem 3.5], it follows that

$$\begin{aligned} &\int_0^t \int_0^s G(X(r))^* S(s-r)^* A^*x^* dW_H(r) ds \\ &= \int_0^t \int_r^t G(X(r))^* S(s-r)^* A^*x^* ds dW_H(r) \\ &= \int_0^t G(X(r))^* S(t-r)^* x^* dW_H(r) - \int_0^t G(X(r))^* x^* dW_H(r) \end{aligned}$$

\mathbb{P} -almost surely.

Plugging these three identities into (3.7) and using that \mathbf{X} is a mild solution, (3.1) follows. \square

Since all terms appearing in (3.1) are almost surely continuous, there is no problem in writing an equation for the stopped process $\langle X(t \wedge \tau), x^* \rangle$ and we did this in the proof of Proposition 3.14. On the other hand, for weakly mild solutions, the integrand in the stochastic integral changes with t , causing problems to obtain an equation for the stopped process. In [3, Appendix], this problem was solved under the assumption that the stochastic convolution is almost surely continuous. However, in the proof of Proposition 3.14, we have shown that for a weak solution, (3.5) holds for all $x^* \in \tilde{E}^*$. Clearly, τ_n can be replaced by an arbitrary stopping time τ . Hence we have

Corollary 3.15. *If \mathbf{X} is a weak (equivalently, weakly mild) solution of (1.1) and τ is a stopping time, then for all $t \geq 0$ and $x^* \in \tilde{E}^*$ we have*

$$\begin{aligned} \langle X(t \wedge \tau), x^* \rangle &= \langle S(t)X(0 \wedge \tau), x^* \rangle + \int_0^t \mathbb{1}_{[0, \tau]} \langle S(t-s)F(X(s)), x^* \rangle ds \\ (3.8) \quad &+ \int_0^t \mathbb{1}_{[0, \tau]} G(X(s))^* S(t-s)^* x^* dW_H(s) . \end{aligned}$$

almost surely.

The question arises whether (3.3) can be extended to hold for all $x^* \in E^*$. This is indeed the case under the following additional assumption.

Hypothesis 3.16. Assume Hypothesis 3.1, that $S(t) \subset \mathcal{L}(\tilde{E}, E)$ for all $t > 0$ and that for $x \in \tilde{E}$ the E -valued map $t \mapsto S(t)x$ is continuous on $(0, \infty)$. Furthermore, assume that for all $t > 0$ the function $(0, t) \ni s \mapsto \|S(s)\|_{\mathcal{L}(\tilde{E}, E)}$ is majorized by a square integrable function.

Assuming Hypothesis 3.16, a slight variation of proof of Lemma 3.13 shows that in this case the integrals in (3.3) are well-defined for $x^* \in E^*$.

Corollary 3.17. *Assume that Hypothesis 3.16 holds. If \mathbf{X} is a weak (equivalently, weakly mild) solution of (1.1), then (3.3) and (3.8) hold for all $x^* \in E^*$.*

Proof. Define

$$V := \{x^* \in E^* : (3.3) \text{ holds a.e.}\} .$$

By Proposition 3.14, $\tilde{E}^* \subset V$ and hence V is weak*-dense in E^* . The claim is proved once we show that V is weak*-closed in E^* . By the Krein-Smulyan theorem (see, e.g., §21.10 (6) of [21]), V is weak*-closed in E^* if and only if $B_V := \{x^* \in V :$

$\{x^* \in E^* : \|x^*\|_{E^*} \leq 1\}$ is weak*-closed in E^* . However, since the weak*-topology is metrizable on bounded sets, it suffices to prove that B_V is sequentially weak*-closed.

Using Hypothesis 3.16, this can be proved similarly as when extending equation (3.5) from $x^* \in D((A^\odot)^2)$ to arbitrary $x^* \in \bar{E}^*$ in the proof of Proposition 3.14. The proof for (3.8) is similar. \square

4. STOCHASTIC INTEGRATION AND MILD SOLUTIONS

We now address the question whether weakly mild (equivalently, weak) solutions of (1.1) satisfy (3.5) also in a ‘strong’ sense, i.e. without testing against functionals x^* . To be more precisely, we want to know whether for all $t \geq 0$ the $\mathcal{L}(H, E)$ -valued process $s \mapsto S(t-s)G(X(s))$ is stochastically integrable (in a sense to be made precise below) and we have, almost surely,

$$(4.1) \quad X(t) = X(0) + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)G(X(s))dW_H(s).$$

We begin by recalling some facts about stochastic integration of operator-valued processes. For time being, B denotes a general separable Banach space and H a separable Hilbert space. We also fix a probability space $(\Omega, \Sigma, \mathbb{P})$ and, defined on this space, an H -cylindrical Wiener process W_H , adapted to a filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$. Fix $T > 0$.

An *elementary process* is a process $\Phi : [0, T] \times \Omega \rightarrow \mathcal{L}(H, B)$ of the form

$$\Phi(t, \omega) = \sum_{n=1}^N \sum_{m=1}^M \mathbb{1}_{(t_{n-1}, t_n] \times A_{mn}}(t, \omega) \sum_{k=1}^K h_k \otimes x_{kmn},$$

where $0 \leq t_0 < \dots < t_N \leq T$, $A_{1n}, \dots, A_{Mn} \in \mathcal{F}_{t_{n-1}}$ are disjoint for all n and the vectors h_1, \dots, h_K are orthonormal in H . If Φ does not depend on ω we also say that Φ is an *elementary function*. For an elementary process, the *stochastic integral* $\int_0^T \Phi(t) dW_H(t)$ is defined by

$$\int_0^T \Phi(t) dW_H(t) := \sum_{n=1}^N \sum_{m=1}^M \mathbb{1}_{A_{mn}} \sum_{k=1}^K [W_H(t_n)h_k - W_H(t_{n-1})h_k] x_{kmn}$$

Now let $\Phi : [0, T] \times \Omega \rightarrow \mathcal{L}(H, B)$ be an H -strongly measurable and adapted process which belongs to $L^2(0, T; H)$ scalarly, i.e. $\Phi^* x^* \in L^0(\Omega; L^2(0, T; H))$ for all $x^* \in B^*$. Then Φ is called *stochastically integrable* (on $(0, T)$) if there exists a sequence Φ_n of elementary processes and an $C([0, T]; E)$ -valued random variable η such that

- (1) $\langle \Phi_n h, x^* \rangle \rightarrow \langle \Phi h, x^* \rangle$ in $L^0(\Omega; L^2(0, T))$ for all $h \in H$ and $x^* \in B^*$ and
- (2) We have

$$\eta(\cdot) = \lim_{n \rightarrow \infty} \int_0^\cdot \Phi_n(t) dW_H(t) \quad \text{in } L^0(\Omega; C([0, T]; B)).$$

In this case, η is called the stochastic integral of Φ and we write $\int_0^t \Phi(t) dW_H(t) := \eta(t)$. In the case where Φ does not depend on ω , we also require that the approximating sequence Φ_n does not depend on ω .

Having defined stochastic integrability, we can now define what we mean by a *mild solution*.

Definition 4.1. A tuple $((\Omega, \Sigma, \mathbb{P}), \mathbb{F}, W_H, \mathbf{X})$ where $(\Omega, \Sigma, \mathbb{P})$ is a probability space endowed with a filtration \mathbb{F} , W_H is an H -cylindrical Wiener process with respect to \mathbb{F} and \mathbf{X} is a continuous, \mathbb{F} -progressive, E -valued process is called a *mild solution* of (1.1) if for all $t \geq 0$ the function $s \mapsto S(t-s)G(X(s))$ is stochastically integrable and (4.1) holds almost surely.

It is clear from the definition of stochastic integrability, that every mild solution of equation $[A, F, G]$ is also a weakly mild solution of $[A, F, G]$ and thus also a weak solution of $[A, F, G]$. Moreover, if \mathbf{X} is a mild solution, then (3.3) even holds for all $x^* \in E^*$ (rather than for $x^* \in \tilde{E}^*$) and the exceptional set outside of which (3.3) holds can be chosen independently of x^* .

The obvious task is to characterize stochastic integrability in terms of the process Φ . Let us first discuss the case of $\mathcal{L}(H, B)$ -valued *functions*, which was considered in [33]. It was proved there that a function $\Phi : [0, T] \rightarrow \mathcal{L}(H, B)$ is stochastically integrable if and only if there exists an B -valued random variable ξ such that

$$(4.2) \quad \langle \xi, x^* \rangle = \int_0^T \Phi(s)^* x^* dW_H(s)$$

if and only if Φ represents a γ -Radonifying operator $R \in \gamma(L^2(0, T; H), B)$. Here, Φ is said to *represent* an operator $R \in \gamma(L^2(0, T; H), B)$ if for all $x^* \in B^*$ the function $t \mapsto \Phi^*(t)x^*$ belongs to $L^2(0, T; H)$ and we have

$$(4.3) \quad \langle Rf, x^* \rangle = \int_0^T [f(t), \Phi^*(t)x^*]_H dt \quad \forall f \in L^2(0, t; H), x^* \in B^*.$$

Note that if Φ is H -strongly measurable, then the operator R is uniquely determined by Φ .

We now recall the definition of γ -Radonifying operators. For more information we refer to the survey article [29]. If \mathcal{H} is a Hilbert space and B is a Banach space, then every finite rank operator $R : \mathcal{H} \rightarrow B$ can be represented in the form $\sum_{n=1}^N h_n \otimes x_n$, where the vectors h_n are orthonormal in \mathcal{H} and the vectors x_n belong to B . For such an operator, we define

$$\|R\|_{\gamma(\mathcal{H}, B)}^2 := \mathbb{E} \left\| \sum_{n=1}^N \gamma_n R h_n \right\|^2.$$

Here, $(\gamma_n)_{n=1}^N$ is a sequence of independent, real-valued standard Gaussian random variables. This defines a norm on the space $\mathcal{H} \otimes B$ of all finite rank operators from \mathcal{H} to E . The completion of $\mathcal{H} \otimes B$ with respect to this norm is denoted by $\gamma(\mathcal{H}, B)$. This space is contractively embedded into $\mathcal{L}(\mathcal{H}, B)$ and an element of $\mathcal{L}(\mathcal{H}, B)$ is called *γ -Radonifying* if it belongs to $\gamma(\mathcal{H}, B)$.

Using the results of [33], we obtain for (1.1) with additive noise:

Proposition 4.2. *Assume Hypotheses 3.1 and 3.16 and that $G \in \mathcal{L}(H, \tilde{E})$ is constant. Then the weak, the weakly mild and the mild solutions of (1.1) coincide. Furthermore, if there exist solutions, the function $s \mapsto S(t-s)G$ represents an element of $\gamma(L^2(0, t; H), E)$ for all $t > 0$.*

Proof. Let \mathbf{X} be a weak (equivalently, a weakly mild) solution of (1.1). If no such solution exists, there is nothing to prove since every mild solution is also a weakly mild solution.

Arguing as in proof of Lemma 3.13, using that as a consequence of Hypothesis 3.16 the map $s \mapsto \langle x, S(t-s)^* x^* \rangle$ is continuous even for $x^* \in E^*$ and $x \in \tilde{E}$, we see that $(s, \omega) \mapsto \langle S(t-s)F(X(s, \omega)), x^* \rangle$ is measurable for all $x^* \in E^*$. By Hypothesis 3.16, $\|S(s)\|_{\mathcal{L}(\tilde{E}, E)}$ is majorized on $(0, t)$ by a square integrable function, say g . Hence, by the boundedness of F on bounded sets we have

$$\|S(t-s)F(X(s, \omega))\| \leq g(t-s) \sup_{r \in (0, t)} \|F(X(r, \omega))\| \in L^1(0, t).$$

This implies that $\int_0^t S(t-s)F(X(s))ds$ can be defined pathwise as an E -valued Bochner integral. Furthermore, this integral is a weakly measurable function of ω . Since E is separable, $\int_0^t S(t-s)F(X(s))ds$ is a strongly measurable function

of ω by the Pettis measurability theorem. Consequently, $\xi := X(t) - S(t)X(0) - \int_0^t S(t-s)F(X(s))ds$ is an E -valued random variable. Since \mathbf{X} is a mild solution, (4.2) holds for all $x^* \in E^*$ by Corollary 3.17. The claim follows from the results of [33]. \square

Let us now return to our discussion of stochastic integrability in a general separable Banach space B . In order to have a powerful integration theory for $\mathcal{L}(H, B)$ -valued processes, we need an additional geometric assumption on B . Of particular importance are the so-called *UMD Banach spaces*. For the definition of UMD spaces and more information, we refer to the survey article [5]. We here confine ourselves to note that every Hilbert space is a UMD space as are the reflexive L^p and Sobolev spaces.

The importance of the UMD property for stochastic integration is that it allows for so-called decoupling, see [13, 26]. Roughly speaking, this allows us to replace the cylindrical Wiener process W_H by an independent copy \tilde{W}_H and thus use the results of [33] pathwise. This program was carried out in [31] and yields a similar characterization of stochastic integrability as in [33] in the case of processes which belong scalarly to $L^p(\Omega; L^2(0, T; H))$. In particular, it is proved there that an H -strongly measurable and adapted process $\Phi : [0, T] \times \Omega \rightarrow \mathcal{L}(H, E)$ which belongs to $L^p(\Omega; L^2(0, T; H))$ scalarly is stochastically integrable if and only if there is a random variable $\xi \in L^p(\Omega; E)$ such that (4.2) holds for all $x^* \in E^*$. This in turn is the case if and only if Φ represents a random variable $R \in L^p(\Omega; \gamma(L^2(0, T; H), E))$. Here ‘represents’ means that (4.3) holds for almost every ω .

A characterization of stochastic integrability for processes Φ which belong scalarly to $L^0(\Omega; L^2(0, T; H))$ is also contained in [31], however, in this characterization one needs information about the whole integral process $\int_0^\cdot \Phi(s)dW_H(s)$; when dealing with weakly mild solutions, such information is not available, whence this characterization cannot be used for our purposes. Therefore, in the proposition below, we use a stopping time argument to reduce to the $L^p(\Omega)$ -case.

Proposition 4.3. *Assume Hypotheses 3.1 and 3.16 and that E is a UMD Banach space. Then the weak, the weakly mild and the mild solutions of (1.1) coincide. Furthermore, if \mathbf{X} is a weak solution, then for all $t \geq 0$ the function $s \mapsto S(t-s)G(X(s))$ represents an element of the space $L^0(\Omega, \gamma(L^2(0, t; H), E))$.*

Proof. Let \mathbf{X} be a weak (equivalently, a weakly mild) solution of (1.1). If no weak solution exists, there is nothing to prove.

For $n \in \mathbb{N}$ and define $\tau_n := \inf\{s > 0 : \|X(s)\| \geq n\}$. Fix $t > 0$. Arguing similar as in the proof of Proposition 4.2, we see that

$$\xi_n := X(t \wedge \tau_n) - S(t)X(0 \wedge \tau_n) - \int_0^t \mathbb{1}_{[0, \tau_n]} S(t-s)F(X(s))ds$$

is a well-defined, bounded, E -valued random variable. It follows from Corollary 3.17, that for $x^* \in E^*$,

$$\langle \xi_n, x^* \rangle = \int_0^t \mathbb{1}_{[0, \tau_n]} G(X(s))^* S(t-s)^* x^* dW_H(s) .$$

almost surely. Since \mathbf{X} has continuous paths and G is bounded on bounded subsets, $\Phi_n : s \mapsto \mathbb{1}_{[0, \tau_n]} S(t-s)G(X(s))$ belongs to $L^\infty(\Omega; L^2(0, t; H))$ scalarly. Hence, by [31, Theorem 5.9], Φ_n is stochastically integrable and

$$(4.4) \quad \begin{aligned} X(t \wedge \tau_n) &= S(t)X(0 \wedge \tau_n) + \int_0^t \mathbb{1}_{[0, \tau_n]} S(t-s)F(X(s))ds \\ &\quad + \int_0^t \mathbb{1}_{[0, \tau_n]} S(t-s)G(X(s))dW_H(s) . \end{aligned}$$

Furthermore, Φ_n represents an element of $L^p(\Omega; \gamma(L^2(0, t; H), E))$ for all $p \geq 1$. Now let N be a set with $\mathbb{P}(N) = 0$ such that for $\omega \notin N$ the map $s \mapsto \Phi_n(s, \omega)$ represents an element $R_n(\omega)$ of $\gamma(L^2(0, t; H), E)$. Such a set exists by [31, Lemma 2.7].

Note that by the continuity of the paths, $\Phi_n(s, \omega) = \Phi(s, \omega) := S(t-s)G(X(s, \omega))$ for all $s \in (0, t)$ and $n \geq n_0 = n_0(\omega)$. Thus, $\Phi(s, \omega)$ represents an element $R(\omega)$ of $\gamma(L^2(0, t; H), E)$ for all $\omega \notin N$. Since $R_n(\omega) \rightarrow R(\omega)$ for all $\omega \notin N$, it follows that R is a strongly measurable $\gamma(L^2(0, t; H), E)$ -valued random variable. Furthermore, R is represented by Φ . By [31, Theorem 5.9], Φ is stochastically integrable and [31, Theorem 5.5] shows that

$$\int_0^t \Phi_n(s) dW_H(s) \rightarrow \int_0^t \Phi(s) dW_H(s) \quad \text{in } L^0(\Omega; E).$$

On the other hand,

$$\xi_n \rightarrow X(t) - S(t)X(0) - \int_0^t S(t-s)F(X(s)) ds$$

pointwise a.e. and hence in $L^0(\Omega; E)$. Thus, letting $n \rightarrow \infty$ in (4.4) finishes the proof. \square

5. AN EQUATION OF REACTION-DIFFUSION TYPE

In this section, we prove well-posedness of the equation (1.2) which we complement with either Dirichlet or Neumann-type boundary conditions. Our state-space E will be $E = C_0(\mathcal{O})$ in the case of Dirichlet boundary conditions and $E = C(\overline{\mathcal{O}})$ in the case of Neumann-type boundary conditions.

We make the following assumptions:

Hypothesis 5.1. (A) \mathcal{O} is an open, bounded domain in \mathbb{R}^d with $C^{2,\alpha}$ -boundary.

The map $a : \overline{\mathcal{O}} \rightarrow \mathbb{R}^{d \times d}$ is symmetric and uniformly strongly elliptic and has entries in $C^{1,\alpha}(\mathcal{O})$.

(F) The function $f : \overline{\mathcal{O}} \times \mathbb{R} \rightarrow \mathbb{R}$ is given as

$$f(x, r) = \sum_{j=0}^{2N+1} b_j(x) r^j$$

where $N \in \mathbb{N}$ and the coefficients b_j belong to $C(\overline{\mathcal{O}})$ and we have $b_{2N+1} \leq -\varepsilon$ for a suitable $\varepsilon > 0$.

(G) The continuous functions $g_k : \overline{\mathcal{O}} \times \mathbb{R} \rightarrow \mathbb{R}$ have the following properties:

(i) There exist $\alpha, \beta \in \ell^2$ such that $|g_k(x, r)| \leq \alpha_k + \beta_k |r|$ for all $x \in \mathcal{O}$ and $r \in \mathbb{R}$.

(ii) There exist continuous functions $\sigma_k : [0, \infty) \rightarrow [0, \infty)$ such that

a) We have $|g_k(x, r_1) - g_k(x, r_2)| \leq \sigma_k(|r_1 - r_2|)$ for all $x \in \mathcal{O}$ and $r_1, r_2 \in \mathbb{R}$.

b) The function $h : (0, \infty) \rightarrow (0, \infty)$, defined as $h(t) := \sum_{k=1}^{\infty} \sigma_k(t)$ is increasing in t and satisfies

$$\int_{0+} h^{-2}(r) dr = \infty.$$

If we impose Dirichlet boundary conditions, then we additionally assume that b_0 (from (F)) belongs to $C_0(\mathcal{O})$ and that $g_k(x, 0) = 0$ for $x \in \partial\mathcal{O}$.

Example 5.2. Let us give an example of functions g_k which satisfy Hypothesis 5.1 (G)(ii). Suppose that $g_k : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ are Hölder continuous of order $\frac{1}{2}$ in the second variable, uniformly in the first, i.e. $c_k = \sup_{x \in \mathcal{O}} \sup_{r_1 \neq r_2} |g_k(x, r_1) - g_k(x, r_2)| \cdot |r_1 - r_2|^{-\frac{1}{2}} < \infty$ for all $k \in \mathbb{N}$. Then part a) of (G)(ii) is satisfied with $\sigma_k(r) = c_k \sqrt{r}$. If, moreover, $(c_k)_{k \in \mathbb{N}} \in \ell^1$, then $h(r) = \|(c_k)\|_{\ell^1} \sqrt{r}$ and also part b) is satisfied.

Before we state the main result of this section, we rewrite equation (1.2) in our abstract form (1.1).

In order to model the differential operator, we follow a variational approach. We consider on $L^2(\mathcal{O})$ the symmetric form

$$\mathfrak{a}[u, v] := \int_{\mathcal{O}} \sum_{i,j=1}^d a_{ij}(x) D_i u(x) D_j v(x) dx$$

endowed with domain either $D(\mathfrak{a}) = H_0^1(\mathcal{O})$ or $D(\mathfrak{a}) = H^1(\mathcal{O})$. Associated to \mathfrak{a} is a self-adjoint operator A_2 on $L^2(\mathcal{O})$ which is the generator of a strongly continuous and analytic semigroup \mathbf{S}_2 , see [37]. The operator A_2 is a second order differential operator endowed with boundary conditions which depend on the choice of $D(\mathfrak{a})$. The choice $D(\mathfrak{a}) = H_0^1(\mathcal{O})$ corresponds to Dirichlet boundary conditions, the choice $D(\mathfrak{a}) = H^1(\mathcal{O})$ corresponds to Neumann-type boundary conditions. To be more precise, in the latter case, we must have that $a \nabla u \cdot \nu = 0$ on $\partial\mathcal{O}$, where ν denotes the unit outer normal to \mathcal{O} . If $a(x) = m(x) \cdot I$, in particular if $a(x) \equiv I$, this reduces to the classical Neumann boundary conditions.

Since the form \mathfrak{a} is in fact sub-Markovian, it follows that \mathbf{S}_2 restricts to a strongly continuous and analytic semigroup \mathbf{S}_p on $L^p(\mathcal{O})$ for all $p \in [2, \infty)$. Its generator A_p is merely the part of A_2 in $L^2(\mathcal{O})$. By [18, Theorems 2.4.2.5 and 2.4.2.6] the domain of A_p is $H^{2,p}(\mathcal{O})$, complemented with either Dirichlet or Neumann boundary conditions. Since \mathbf{S}_p is analytic, it maps into $D(A_p^\infty)$, in particular, $E := C_0(\mathcal{O})$ (in the case of Dirichlet boundary conditions) resp. $E = C(\overline{\mathcal{O}})$ (in the case of Neumann boundary conditions) is invariant under \mathbf{S}_p and \mathbf{S}_p restricts to a strongly continuous semigroup on E whose generator A_E is the part of A_2 in E .

In Hypothesis 3.1, we choose $E = C_0(\mathcal{O})$ or $E = C(\overline{\mathcal{O}})$, depending on the boundary conditions, $\tilde{E} = L^p(\mathcal{O})$ for some $p \geq 2$ and $A = A_p$. We note that the complex interpolation space $[D(A_p), L^p(\mathcal{O})]_\theta$ is isomorphic to a subspace of $H^{2\theta,p}(\mathcal{O})$. Hence, using the relationship between fractional domain spaces and complex interpolation spaces and the fact that $\|S_p(t)\|_{\mathcal{L}(L^p(\mathcal{O}), D((w-A_p)^\theta))} \lesssim t^{-\theta}$, it follows from Sobolev embedding that also Hypothesis 3.16 is satisfied if θ and p are chosen large enough in relation to d .

In the proof of uniqueness of equation (1.2), we will use the following result about the resolvents which follows from the regularity assumptions on a and \mathcal{O} and [17, Theorem 6.14] resp. [17, Theorem 6.31]:

Lemma 5.3. *If $f \in C^\alpha(\overline{\mathcal{O}})$, then $R(\lambda, A_p)f = R(\lambda, A_E) \in C^{2,\alpha}(\overline{\mathcal{O}})$. In particular, the second order derivatives of $R(\lambda, A_p)f$ are bounded.*

Let us next turn to the nonlinearity f . We define $F : C(\overline{\mathcal{O}}) \rightarrow C(\overline{\mathcal{O}}) \hookrightarrow \tilde{E}$ by

$$[F(u)](x) := f(x, u(x)).$$

Then F is locally Lipschitz continuous from $C(\overline{\mathcal{O}})$ to $C(\overline{\mathcal{O}})$. Indeed, by the boundedness of the coefficients b_j , it follows that $\frac{\partial}{\partial r} f(x, r)$ is bounded on bounded subsets of \mathbb{R} , uniformly in $x \in \mathcal{O}$. Now the mean value theorem immediately yields the local Lipschitz continuity. In particular, F satisfies (1) of Hypothesis 3.1. Note that in the case of Dirichlet boundary conditions F maps $C_0(\mathcal{O})$ into itself by the additional assumption $b_0 \in C_0(\mathcal{O})$

It is easy to see that, for suitable constants $a_1, a_2, b_1, b_2 \in \mathbb{R}$, we have

$$a_1 - b_1 r^{2N+1} \leq f(x, r) \leq a_2 - b_2 r^{2N+1}.$$

This yields, see [23, Example 4.5], that for $u, v \in E$ and $u^* \in \partial\|u\|$, the subdifferential of the norm at u , we have

$$(5.1) \quad \langle F(u+v) - F(v), u^* \rangle \leq a(1 + \|v\|)^{2N+1} - b\|u\|^{2N+1}.$$

This dissipativity property plays an important role in proving existence of (global) solutions for (1.2), see Section 5.2.

To model the stochastic term, we put $H = \ell^2$. Denoting the canonical orthonormal basis of ℓ^2 by (e_k) , the operator $W_H : L^2(\mathbb{R}_+; \ell^2) \rightarrow L^2(\Omega)$, defined by

$$W_H(f) := \sum_{k=1}^{\infty} [f(t), e_k] dw_k(t)$$

is an H -cylindrical Wiener process, cf. [29].

We define $G : E \rightarrow \mathcal{L}(\ell^2, L^p(\mathcal{O}))$ by

$$G(u)h := \sum_{k=1}^{\infty} h_k g_k(\cdot, u(\cdot)).$$

Note that for every $k \in \mathbb{N}$ and $u \in E$ we actually have $g_k(\cdot, u(\cdot)) \in C(\overline{\mathcal{O}}) \subset L^p(\mathcal{O})$ with $\|g(\cdot, u(\cdot))\|_{\infty} \leq \alpha_k + \beta_k \|u\|_{\infty}$ so that the above series converges in $L^p(\mathcal{O})$. In fact, even the series

$$\sum_{k=1}^{\infty} |G(u)e_k|^2 = \sum_{k=1}^{\infty} |g_k(\cdot, u(\cdot))|^2$$

converges in $L^p(\mathcal{O})$. Hence, by [32, Lemma 2.1], $G(u) \in \gamma(\ell^2, L^p(\mathcal{O}))$ and

$$\|G(u)\|_{\gamma(\ell^2, L^p(\mathcal{O}))} \simeq \left\| \left(\sum_{k=1}^{\infty} |g_k(\cdot, u(\cdot))|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{O})} \leq |\mathcal{O}|(\|\alpha\|_2 + \|\beta\|_2 \|u\|_{\infty}).$$

This proves that $G : E \rightarrow \gamma(H, \tilde{E})$ has linear growth. Let us next prove that G is continuous. To that end, let $u_n \rightarrow u$ in $C_0(\mathcal{O})$. Then $C := \sup_{n \in \mathbb{N}} \|u_n\|_{\infty} < \infty$. Employing [32, Lemma 2.1] a second time, we see that

$$\|G(u_n) - G(u)\|_{\gamma(\ell^2, L^p(\mathcal{O}))} \simeq \left\| \left(\sum_{k=1}^{\infty} |g_k(\cdot, u_n(\cdot)) - g_k(\cdot, u(\cdot))|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{O})}.$$

But, by dominated convergence, the latter converges to 0, since the integrands are dominated by the bounded function $2(\|\alpha\|_{\ell^2} + \|\beta\|_{\ell^2} C)$ and converges to 0 pointwise.

Having clarified the interpretation of equation (1.2), we can now formulate the main result of this section.

Theorem 5.4. *Assume Hypothesis 5.1. Then for every initial distribution $\mu \in \mathcal{P}(E)$, there exists a unique (in law) mild solution u of equation (1.2) such that $u(0) = u_0$ has distribution μ . Moreover, pathwise uniqueness holds for (1.2), all solutions are strong Markov processes with respect to a common transition semigroup \mathbf{T} and, if \mathbf{P}_x denotes the distribution of the solution of equation (1.2) with initial value $x \in E$. Then the map $x \mapsto \mathbf{P}_x$ is continuous; in particular, \mathbf{T} is a strictly continuous Feller semigroup.*

The proof of Theorem 5.4 is given in the three following Subsections. We first prove pathwise uniqueness of solutions, then existence of solutions for deterministic initial values $u_0 \in E$ and then use our abstract results to finish the proof.

5.1. Proof of uniqueness. We first prove pathwise uniqueness for equation (1.2). In view of Theorem 3.11, this proves uniqueness in law for (1.2).

Proof. We need some preparation.

We may (and shall) assume that $h(x) \geq \sqrt{x}$, otherwise replacing $h(x)$ with $h(x) + \sqrt{x}$. Similar as in the proof of the classical Yamada-Watanabe theorem [44], we choose a decreasing sequence $a_n \downarrow 0$ such that $a_0 = 1$ and

$$\int_{a_n}^{a_{n-1}} h^{-2}(r) dr = n.$$

Next pick functions $\psi_n \in C_c^\infty(\mathbb{R})$ such that $\text{supp } \psi_n \subset (a_n, a_{n-1})$ and

$$0 \leq \psi_n(r) \leq \frac{2}{nh^2(r)} \leq \frac{2}{nr} \quad \forall r \in \mathbb{R} \quad \text{and} \quad \int_{\mathbb{R}} \psi_n(r) dr = 1 .$$

We now define

$$\phi_n(r) := \int_0^{|r|} \int_0^s \psi_n(\tau) d\tau ds .$$

It follows from the choice of the ψ_n that every function ϕ_n vanishes in a neighborhood of zero, namely in $(-a_n, a_n)$. Consequently, $\phi_n \in C^\infty(\mathbb{R})$. More precisely,

$$\phi'_n(r) = \text{sgn } r \int_0^{|r|} \psi_n(\tau) d\tau \quad \text{and} \quad \phi''_n(r) = \psi_n(|r|) .$$

We note that

$$|r| - a_{n-1} = \int_0^r \mathbb{1}_{(a_{n-1}, \infty)}(s) ds \leq \phi_n(r) \leq |r| ,$$

implying that $\phi_n(r) \rightarrow |r|$ uniformly on \mathbb{R} . Moreover, $\phi'_n(r)f = |r| \int_0^{|r|} \psi_n(\tau) d\tau$ converges to $|r|$ pointwise.

To prove pathwise uniqueness, let u_1, u_2 be two weak solutions of (1.2), defined on the same probability space and with respect to the same sequence of Brownian motions (w_k) . We furthermore assume that $u_1(0) = u_2(0)$ almost surely. We have to prove that $u_1 = u_2$ almost surely. For $m \in \mathbb{N}$, we define the stopping time τ_m by $\tau_m := \inf\{t > 0 : \|u_1(t)\| \vee \|u_2(t)\| > m\}$.

Now fix $\lambda > 0$. Then $x^* := \lambda R(\lambda, A_E)^* \delta_x \in D(A^*)$. Since u_1 and u_2 are weak solutions, (3.1) yields

$$\begin{aligned} & [\lambda R(\lambda, A_E)(u_1(t \wedge \tau_m) - u_2(t \wedge \tau_m))](x) \\ &= \int_0^{t \wedge \tau_m} [A_E \lambda R(\lambda, A_E)(u_1(s) - u_2(s))](x) ds \\ &+ \int_0^{t \wedge \tau_m} [\lambda R(\lambda, A_E)(F(u_1(s)) - F(u_2(s)))](x) ds \\ &+ \sum_{k=1}^{\infty} \int_0^t \mathbb{1}_{[0, \tau_m]}(s) [\lambda R(\lambda, A_E)(g_k(u_1(s)) - g_k(u_2(s)))](x) dw_k(s) . \end{aligned}$$

To simplify notation, let us write $\delta_\lambda^m u(s) := \lambda R(\lambda, A_E)(u_1(s \wedge \tau_m) - u_2(s \wedge \tau_m))$. We also abbreviate $\delta_\lambda F(s) := \lambda R(\lambda, A_E)(F(u_1(s \wedge \tau_m)) - F(u_2(s \wedge \tau_m)))$ and $\delta_\lambda^m g_k(s) := \lambda R(\lambda, A_E)(g_k(u_1(s \wedge \tau_m)) - g_k(u_2(s \wedge \tau_m)))$.

Now, by Itô's formula,

$$\begin{aligned} \phi_n(\delta_\lambda^m u(t)(x)) &= \int_0^{t \wedge \tau_m} \phi'_n(\delta_\lambda^m u(t)(x)) [A_E \delta_\lambda^m u(s)(x) + \delta_\lambda^m F(s)(x)] ds \\ &+ \frac{1}{2} \int_0^{t \wedge \tau_m} \phi''_n(\delta_\lambda^m u(s)(x)) \sum_{k,l=1}^{\infty} \delta_\lambda^m g_k(s)(x) \delta_\lambda^m g_l(s)(x) ds + \text{a martingale} , \end{aligned}$$

for all $x \in \mathcal{O}$.

Next, let $0 \leq v \in D(A_2) \cap C_b^2(\mathcal{O})$; below we will take $v = R(\mu, A_2)\varphi$ for a function $\varphi \in C_c^\infty(\mathcal{O})$. Note that since A_2 is selfadjoint and the semigroups \mathbf{S}_p are consistent, it follows that $v \in D(A_p^*)$ and $A_p^* v = A_2 v$. Integrating against v and

taking expectations, we obtain

$$\begin{aligned}
\mathbb{E}\langle \phi_n(\delta_\lambda u(t)), v \rangle &= \mathbb{E} \int_0^{t \wedge \tau_m} \langle \phi'_n(\delta_\lambda^m u(s)) A_{C_0} \delta_\lambda u(s), v \rangle ds \\
&+ \mathbb{E} \int_0^{t \wedge \tau_m} \langle \phi'_n(\delta_\lambda^m u(s)) \delta_\lambda^m F(s), v \rangle ds \\
&+ \frac{1}{2} \mathbb{E} \int_0^{t \wedge \tau_m} \left\langle \psi_n(|\delta_\lambda^m u(s)|) \left(\sum_{k=1}^{\infty} \delta_\lambda^m g_k(s) \right)^2, v \right\rangle ds \\
&=: I_1(n, \lambda) + I_2(n, \lambda) + I_3(n, \lambda) .
\end{aligned}$$

We now proceed in several steps.

Step 1: We provide an estimate for $I_1(n, \lambda)$.

This part of the proof is similar to that of [27, Lemma 2.2]. First consider arbitrary $u, v \in D(A_2)$. Integrating by parts, using the boundary conditions, we obtain

$$\begin{aligned}
\int_{\mathcal{O}} \phi'_n(u(x)) A u(x) v(x) dx &= \int_{\mathcal{O}} \phi'_n(u(x)) u(x) A v(x) dx \\
&+ \int_{\mathcal{O}} \phi''_n(u(x)) \langle a(x) \nabla u(x), \nabla v(x) \rangle_{\mathbb{R}^d} u(x) dx \\
&- \int_{\mathcal{O}} \phi''_n(u(x)) \langle a(x) \nabla u(x), \nabla u(x) \rangle_{\mathbb{R}^d} v(x) dx \\
&=: J_1(n) + J_2(n) - J_3(n)
\end{aligned}$$

Next we additionally assume that $v \geq 0$. We define $\tilde{u}(x) := a(x)^{\frac{1}{2}} \nabla u(x)$ and $\tilde{v}(x) := a(x)^{\frac{1}{2}} \nabla v(x)$. In what follows, an index k indicates the k -th component of a vector. Noting that $\phi''_n(r) = \psi_n(|r|) \geq 0$, we estimate

$$\begin{aligned}
J_2(n) - J_3(n) &= \sum_{k=1}^d \int_{\mathcal{O}} \psi_n(|u(x)|) [\tilde{u}_k(x) \tilde{v}_k(x) u(x) - \tilde{u}_k(x)^2 v(x)] dx \\
&\leq \sum_{k=1}^d \int_{B_k} \psi_n(|u(x)|) [\tilde{u}_k(x) \tilde{v}_k(x) u(x) - \tilde{u}_k(x)^2 v(x)] dx \\
&\leq \sum_{k=1}^d \int_{B_k} \psi_n(|u(x)|) \tilde{u}_k(x) \tilde{v}_k(x) u(x) dx ,
\end{aligned}$$

where

$$B_k := \{x : \tilde{u}_k(x)^2 v(x) \leq \tilde{u}_k(x) \tilde{v}_k(x) u(x)\} \cap \{x : v(x) > 0\} .$$

Let us also define

$$B_k^\pm := B_k \cap \{x : \tilde{u}_k(x) \geq 0\} \quad \text{and} \quad B_k^0 := B_k \cap \{x : \tilde{u}_k(x) = 0\} .$$

On B_k^+ we have $0 < \tilde{u}_k(x) v(x) \leq \tilde{v}_k(x) u(x)$ and, consequently,

$$\int_{B_k^+} \psi_n(|u(x)|) \tilde{u}_k(x) \tilde{v}_k(x) u(x) dx \leq \int_{B_k^+} \psi_n(|u(x)|) u(x)^2 \tilde{v}_k(x)^2 v(x)^{-1} dx .$$

Similarly, on B_k^- we have $0 > \tilde{u}_k(x) v(x) \geq \tilde{v}_k(x) u(x)$, yielding also

$$\int_{B_k^-} \psi_n(|u(x)|) \tilde{u}_k(x) \tilde{v}_k(x) u(x) dx \leq \int_{B_k^+} \psi_n(|u(x)|) u(x)^2 \tilde{v}_k(x)^2 v(x)^{-1} dx .$$

We note that this estimate also holds for the integrals over B_k^0 , since in that case the integral on the left-hand side vanishes whereas the integral on the right-hand

side is nonnegative. Summing up, using the properties of ψ_n , we find

$$\begin{aligned} J_2(n) - J_3(n) &\leq \int_{\{v>0\}} \psi_n(|u(x)|)(u(x))^2 \frac{\|a(x)^{\frac{1}{2}} \nabla v(x)\|^2}{v(x)} dx \\ &\leq \int_{\{v>0\}} \frac{2}{n} \mathbb{1}_{\{a_{n-1} \leq |u(x)| \leq a_n\}}(x) |u(x)| \frac{\|a(x)^{\frac{1}{2}}\|^2 \|\nabla v(x)\|^2}{v(x)} dx \\ &\leq \frac{2a_n M}{n} \int_{\{|v|>0\}} \frac{\|\nabla v(x)\|^2}{v(x)} dx. \end{aligned}$$

We note that if $v \in C_b^2(\mathcal{O})$, then, as a consequence of [27, Lemma 2.1], the integral in the above equation is finite.

Step 2: We take limits as $\lambda \rightarrow \infty$.

By Step 1, we have

$$\begin{aligned} \mathbb{E} \langle \phi_n(\delta_\lambda^m u(t)), v \rangle &\leq C(v) \frac{1}{n} + \mathbb{E} \int_0^{t \wedge \tau_m} \langle \phi'_n(\delta_\lambda^m u(s)) \delta_\lambda^m u(s), A_2 v \rangle ds \\ &\quad + \mathbb{E} \int_0^{t \wedge \tau_m} \langle \phi'_n(\delta_\lambda^m u(s)) \delta_\lambda^m F(s), v \rangle ds \\ &\quad + \frac{1}{2} \mathbb{E} \int_0^{t \wedge \tau_m} \left\langle \psi_n(|\delta_\lambda^m u(s)|) \left(\sum_{k=1}^{\infty} \delta_\lambda^m g_k(s) \right)^2, v \right\rangle ds. \end{aligned}$$

Here, $C(v)$ is a constant depending on v . For $\phi \in E$ we have $\lambda R(\lambda, E)\phi \rightarrow \phi$ uniformly on $\overline{\mathcal{O}}$. Hence, upon $\lambda \rightarrow \infty$ we find

$$\begin{aligned} \mathbb{E} \langle \phi_n(\delta^m u(t)), v \rangle &\leq C(v) \frac{1}{n} + \mathbb{E} \int_0^{t \wedge \tau_m} \langle \phi'_n(\delta^m u(s)) \delta^m u(s), A v \rangle ds \\ &\quad + \mathbb{E} \int_0^{t \wedge \tau_m} \langle \phi'_n(\delta u(s)) \delta^m f(s), v \rangle ds \\ &\quad + \frac{1}{2} \mathbb{E} \int_0^{t \wedge \tau_m} \left\langle \psi_n(|\delta^m u(s)|) \left(\sum_{k=1}^{\infty} \delta^m G_k(s) \right)^2, v \right\rangle ds, \\ &=: C(v) \frac{1}{n} + I_1(n) + I_2(n) + I_3(n) \end{aligned}$$

where we have used the abbreviations $\delta^m u(s) := u_1(s \wedge \tau_m) - u_2(s \wedge \tau_m)$, $\delta^m f(s) := f(u_1(s \wedge \tau_m)) - f(u_2(s \wedge \tau_m))$ and $\delta^m g_k(s) := g_k(u_1(s \wedge \tau_m)) - g_k(u_2(s \wedge \tau_m))$.

Step 3: We estimate $I_2(n)$ and $I_3(n)$ and let $n \rightarrow \infty$.

By construction, $\psi_n(|\delta^m u(s, x)|) \leq \frac{2}{n} h(|\delta^m u(s, x)|)^{-2}$. Furthermore, by assumption (G)(ii), we have

$$\left(\sum_{k=1}^{\infty} \delta^m g_k(s, x) \right)^2 \leq \left(\sum_{k=1}^{\infty} \sigma_k(|\delta^m u(s, x)|) \right)^2 = h^2(\delta^m u(s, x)).$$

Hence, since v is positive and $\psi_n(r) \leq \frac{1}{nh^2(r)}$, we may conclude that $I_3(n) \leq \frac{t \wedge \tau_m}{2n} \|v\|_{L^1(\mathcal{O})}$.

Concerning $I_2(n)$, note that since f is locally Lipschitz continuous in the second variable, uniformly in the first and since $|\delta^m u(s, x)| \leq 2m$ for all $s \in [0, t]$ and $x \in \mathcal{O}$, we have $|\phi_n(\delta^m u(s, x)) \delta^m f(s, x)| \leq L_m |\phi_n(\delta^m u(s, x))| |\delta^m u(s, x)|$.

Since $\phi_n(r) \uparrow r$ uniformly for $r \in \mathbb{R}$ and $\phi'_n(r)r \rightarrow |r|$ we find upon letting $n \rightarrow \infty$ that

$$\mathbb{E} \langle |\delta^m u(t)|, v \rangle \leq \mathbb{E} \int_0^{t \wedge \tau_m} \langle |\delta^m u(s)|, A_2 v \rangle ds + L_m \mathbb{E} \int_0^{t \wedge \tau_m} \langle |\delta^m u(s)|, v \rangle ds$$

for all $0 \leq v \in D(A_2) \cap C_b^2(\mathcal{O})$. Setting $\Phi_m(t) := \mathbb{E}|\delta^m u(t)|$ and using Fubini's theorem, it follows that

$$(5.2) \quad \langle \Phi_m(t), v \rangle \leq \int_0^t \langle \Phi_m(s), Av + L_m v \rangle \, ds.$$

Step 4: We finish the proof.

We now choose $0 \leq \varphi \in C_c^\infty(\mathcal{O})$ and put $v = R(\mu, A_2)\varphi \in D(A_2)$ in the previous arguments. Note that by Lemma 5.3 we indeed have $v \in C_b^2(\mathcal{O})$, so that (5.2) holds for this function v by the previous steps. Since $A_2 R(\mu, A_2) = \mu R(\mu, A_2) - I$, we have

$$\begin{aligned} \langle R(\mu, A_2)\Phi_m(t), \varphi \rangle &\leq \int_0^t \langle \Phi_m(s), \mu R(\mu, A_2)\varphi - \varphi \rangle \, ds + L_m \int_0^t \langle R(\mu, A_2)\Phi_m(s), \varphi \rangle \, ds \\ &\leq (\mu + L_m) \int_0^t \langle R(\mu, A_2)\Phi_m(s), \varphi \rangle \, ds. \end{aligned}$$

Consequently, by the Gronwall lemma, $\langle R(\mu, A_2)\Phi_m(t), \varphi \rangle \equiv 0$. Since $0 \leq \varphi \in C_c^\infty(\mathcal{O})$ was arbitrary, we conclude that $R(\mu, A_2)\Phi(t) \equiv 0$ and thus $\Phi(t) \equiv 0$. This in turn yields $u_1(t, x) = u_2(t, x)$ for all $0 \leq t \leq \tau_m$ and all $x \in \mathcal{O}$. Since $\tau_m \uparrow \infty$ almost surely, it follows that, almost surely, $u_1(t) = u_2(t)$ for all $t \geq 0$ and finishes the proof of pathwise uniqueness. \square

5.2. Proof of existence. We now proceed to prove existence of solutions to equation (1.2) with deterministic initial values. If the map $G(\cdot) : E \rightarrow \mathcal{L}(\ell^2, L^p(\mathcal{O}))$ is additionally assumed to be bounded, then existence of (mild) solutions to (1.2) follows directly from the results of [2]. It thus remains to extend these results to maps G of linear growth. To do so, we employ Lemma 3.9.

For $n, k \in \mathbb{N}$ we define

$$g_k^{(n)}(x, t) := \begin{cases} g(x, t), & \text{if } |t| \leq n \\ g(x, n), & \text{if } t > n \\ g(x, -n), & \text{if } t < -n. \end{cases}$$

We note that for every n , the sequence $(g_k^{(n)})_{k \in \mathbb{N}}$ satisfies Hypothesis 5.1, more precisely, for all $n \in \mathbb{N}$ the same vectors $\alpha, \beta \in \ell^2$ and the same functions σ_k and h can be used for the sequence $(g_k^{(n)})_{k \in \mathbb{N}}$. Moreover, the associated operators $G^{(n)}(u)v := \sum_{k=1}^\infty v_k g_k^{(n)}(\cdot, u(\cdot))$ take values in $\gamma(\ell^2, L^p(\mathcal{O}))$ and are *bounded* on E . Of course these operators are not uniformly bounded, but the $G^{(n)}$ are uniformly of linear growth.

By the results of [2], for any $u_0 \in E$, there exists a mild (hence weak) solution u_n of equation $[A, F, G]$ with initial datum u_0 . Consequently, by Theorem 3.5, there exists a solution \mathbf{P}_n of the local martingale problem for $\mathcal{L}_{[A, F, G^{(n)}]}$ with initial distribution δ_{u_0} . Noting that $G_n(\cdot)$ converges to $G(\cdot)$ in $\mathcal{L}(\ell^2, L^p(\mathcal{O}))$, uniformly on the compact subsets of E (even on the bounded subsets of E), existence of a solution to the local martingale problem for $\mathcal{L}_{[A, F, G]}$ with initial distribution δ_{u_0} follows from Lemma 3.9, once we proved that the sequence \mathbf{P}_n is tight.

It remains to prove that the measures \mathbf{P}_n are tight. We shall prove this using the factorization method [9]. More precisely, we will proceed in two steps. First, we prove that the solutions u_n are uniformly bounded in probability and then we apply the factorization method in a second step.

Proof. We use the notation introduced above, i.e. for fixed $u_0 \in E$, u_n denotes a mild solution, defined on a probability space $(\Omega_n, \Sigma_n, \mathbb{P}_n)$, of equation $[A, F, G]$ with initial datum u_0 and $\mathbf{P}_n \in \mathcal{P}(C([0, \infty); E))$ denotes its distribution. W_H^n denotes the cylindrical Wiener process on Ω_n with respect to which u_n is a mild solution.

Note that, by the previous subsection and [2], equation $[A, F, G_n]$ is well-posed, so that \mathbf{P}_n does not depend on the particular mild solution u_n .

Step 1: We prove that the solutions u_n are uniformly bounded in probability, more precisely, for all $q > 2$ and $T > 0$ there exists a constant $C = C(T, q)$, independent of n , such that

$$\mathbb{E}_n \|u\|_{C([0, T]; E)}^q \leq C(1 + \|u_0\|)^q$$

for all $n \in \mathbb{N}$. Here, \mathbb{E}_n denotes expectation with respect to \mathbb{P}_n .

The following is [23, Lemma 4.8]. It is the key-lemma to prove boundedness in probability.

Lemma 5.5. *Let A be the generator of a strongly continuous contraction semigroup S on E and $F : E \rightarrow E$ satisfy (5.1). If $\varphi, \psi \in C([0, T]; E)$ satisfy*

$$u(t) = \int_0^t S(t-s)F(u(s) + v(s)) ds \quad \forall t \in [0, T]$$

then

$$\sup_{t \in [0, T]} \|v(t)\| \leq \left(\frac{4a}{b}\right)^{\frac{1}{2N+1}} (1 + \sup_{t \in [0, T]} \|v(t)\|).$$

Using Lemma 5.5, the proof of boundedness in probability follows the lines of [23, Theorem 4.9].

We fix $k \in \mathbb{N}$ and define $\tau_k^n : \Omega_n \rightarrow [0, \infty)$ by $\tau_k^n := \inf\{t > 0 : \|u_n(t)\| > k\}$ with the convention that $\inf \emptyset = 0$. We now define $u_k^n(t) := u_n(t \wedge \tau_k^n)$. By Corollary 3.15, we have

$$\begin{aligned} u_k^n(t) &= S(t \wedge \tau_k^n)u_0 + \int_0^{t \wedge \tau_k^n} S(t-s)F(u_k^n(s)) ds \\ &\quad + \int_0^t \mathbb{1}_{[0, \tau_k^n]}(s)S(t-s)G_n(u_k^n(s)) dW_H^n(s). \end{aligned}$$

Applying Lemma 5.5 pathwise with

$$\begin{aligned} u(t) &= u_k^n(t) - S(t \wedge \tau_k^n)u_0 - \int_0^t \mathbb{1}_{[0, \tau_k^n]}(s)S(t-s)G_n(u_k^n(s)) ds \\ v(t) &= S(t \wedge \tau_k^n)u_0 + \int_0^t \mathbb{1}_{[0, \tau_k^n]}(s)S(t-s)G_n(u_k^n(s)) dW_H^n(s), \end{aligned}$$

which are continuous on $[0, \tau_k^n]$, we obtain

$$\begin{aligned} &\mathbb{E}_n \sup_{t \in [0, T]} \left\| \int_0^{t \wedge \tau_k^n} S(t-s)F(u_k^n(s)) ds \right\|^p \\ &\leq C \left(1 + \|u_0\|^p + \mathbb{E}_n \sup_{t \in [0, T]} \left\| \int_0^t \mathbb{1}_{[0, \tau_k^n]}(s)S(t-s)G_n(u_k^n(s)) dW_H^n(s) \right\|^p \right), \end{aligned}$$

where C is a constant independent of n and k .

We next prove an estimate for the stochastic term. To that end, pick θ such that $D((w - A_p)^\theta) \hookrightarrow E$. Then

$$\begin{aligned} &\mathbb{E}_n \sup_{t \in [0, T]} \left\| \int_0^t \mathbb{1}_{[0, \tau_k^n]}(s)S(t-s)G_n(u_k^n(s)) dW_H(s) \right\|_E^q \\ &\lesssim \mathbb{E}_n \sup_{t \in [0, T]} \left\| \int_0^t \mathbb{1}_{[0, \tau_k^n]}(s)S(t-s)G_n(u_k^n(s)) dW_H(s) \right\|_{D((w - A_p)^\theta)}^q \\ &\lesssim \int_0^T \mathbb{E}_n \|s \mapsto (t-s)^{-r} \mathbb{1}_{[0, \tau_k^n]}(s)G_n(u_k^n(s))\|_{\gamma(L^2(0, t; \ell^2), L^p(\mathcal{O}))}^q ds, \end{aligned}$$

where we have used [32, Proposition 4.2] in the last estimate. Here, $r \in (0, \frac{1}{2})$ is such that $\theta < r - \frac{1}{q}$. Note that with appropriate choices of θ and p , it can always be arranged that there exists such an r .

Since $L^p(\mathcal{O})$ has type 2, $L^2(0, t; \gamma(\ell^2, L^p(\mathcal{O}))) \hookrightarrow \gamma(L^2(0, t; \ell^2), L^p(\mathcal{O}))$, see [29, Theorem 11.6]. Thus the above may be estimated by

$$\begin{aligned} &\lesssim \int_0^T \left(\int_0^t (t-s)^{-2r} \mathbb{E}_n \|G_n(u_k^n(s))\|_{\gamma(\ell^2, L^p(\mathcal{O}))}^2 ds \right)^{\frac{q}{2}} dt \\ &\lesssim \int_0^T \left(\int_0^t (t-s)^{-2r} (\|\alpha\|_2^2 + \|\beta\|_2^2 \mathbb{E}_n \|u_k^n(s)\|_E^2) ds \right)^{\frac{q}{2}} dt \\ &\lesssim \left(\int_0^T s^{-2r} ds \right)^{\frac{q}{2}} \int_0^T \|\alpha\|_2^q + \|\beta\|_2^q \mathbb{E}_n \|u_k^n(s)\|_E^q ds \end{aligned}$$

where we have used the uniform linear growth of the G_n in the second and Young's inequality in the third line. Collecting the estimates, we find that

$$\mathbb{E}_n \sup_{t \in [0, T]} \|u_n^k(t)\|_E^q \leq C_1(1 + \|u_0\|_E^q) + C_2(T) \mathbb{E}_n \sup_{t \in [0, T]} \|u_n^k(t)\|_E^q$$

where C_1 and $C_2(T)$ are certain constants, independent of n and k and $C_2(T) \rightarrow 0$ as $T \rightarrow 0$. Thus, for T_0 small enough

$$\mathbb{E}_n \sup_{t \in [0, T_0]} \|u_n^k(t)\|_E^q \leq (1 - C_2(T_0))^{-1} C_1(1 + \|u_0\|_E^q)$$

for all $n, k \in \mathbb{N}$. For $k \rightarrow \infty$ we obtain, by Fatou's Lemma, that u_n belongs to $L^q(\Omega_n; C([0, T_0]; E))$ with

$$\mathbb{E}_n \|u_n\|_{L^q(\Omega; C([0, T_0]; E))}^p \leq (1 - C_2(T_0))^{-1} C_1(1 + \|u_0\|_E^q)$$

where the constants (in particular the constant T_0) do not depend on n .

To extend this from T_0 to general T , let us first note that since equation $[A, F, G_n]$ is well-posed, it is completely well-posed. In particular, for every initial distribution there exists a unique (in probability) weak solution with that initial distribution. One can now repeat the above computation, to prove that the above estimate generalizes to initial values $u_0 \in L^q(\Omega_n; E)$. Here, of course, we have to replace $\|u_0\|_E^q$ with $\mathbb{E}_n \|u_0\|_E^q$. In particular, one can prove that $S(t - \cdot)G(u_n(\cdot)) \in L^2(0, t; \gamma(\ell^2, \mathcal{D}((w - A_p)^\theta)))$, so that this function is stochastically integrable with values in $\mathcal{D}((w - A_p)^\theta)$ and hence, trivially, with values in E . This implies that u_n is a mild solution.

Using uniqueness of solutions, we can now iterate this as usual and, given $T > 0$, obtain a constant $C = C(T, p)$ such that

$$\mathbb{E}_n \|u_n\|_{L^q(\Omega; C([0, T]; E))}^q \leq C(1 + \mathbb{E}_n \|u_0\|_E^q).$$

This finishes the proof of Step 1.

Step 2: We apply the factorization method.

For $\gamma \in (0, 1]$, the factorization operator R_γ is defined on $L^q([0, T]; E)$ (or on $L^q([0, T]; L^p(\mathcal{O}))$) by

$$[R_\gamma f](t) := \int_0^t (t-s)^{\gamma-1} S(t-s) f(s) ds.$$

Smoothing properties of these operators were studied in [9], its use to prove tightness of stochastic processes goes back to [16]. The essential property, see [15, Proposition 1], is that if the semigroup \mathbf{S} is immediately compact (note that this is fulfilled in our situation) and $\gamma \in (q^{-1}, 1]$, then R_γ defines a compact operator from $L^q([0, T]; E)$ to $C([0, T]; E)$.

We now prove tightness of the measures (\mathbf{P}_n) . Using [19, Proposition 16.6], we see that it suffices to prove that the restrictions $\mathbf{P}_n^{[0, T]}$ of \mathbf{P}_n to $C([0, T]; E)$ are tight.

For the rest of the proof, we fix $T > 0$ and $\varepsilon > 0$ be given. We need to produce a compact set $\mathcal{K} \subset C([0, T]; E)$ such that $\mathbf{P}_n^{[0, T]}(\mathcal{K}) \geq 1 - \varepsilon$ for all $n \in \mathbb{N}$.

By Step 1 and Chebyshev's inequality, we find a bounded set $\mathcal{B} \subset C([0, T]; E)$ such that $\mathbb{P}_n(u_n|_{[0, T]} \in \mathcal{B}) \geq 1 - \varepsilon/2$. Since F is bounded on bounded subsets of E , there is a bounded set $\mathcal{B}_F \subset C([0, T]; E)$ such that $\mathbb{P}_n(F \circ u_n|_{[0, T]} \in \mathcal{B}_F) \geq 1 - \varepsilon/2$. Since R_1 is compact, the set $\mathcal{K}_F := R_1\mathcal{B}_F$ is compact. Furthermore, $\mathbb{P}_n(R_1F \circ u_n|_{[0, T]} \in \mathcal{K}_F) \geq 1 - \varepsilon$. Thus, with probability at least $1 - \varepsilon/2$, the deterministic convolution

$$t \mapsto \int_0^t S(t-s)F(u_n(s))ds$$

takes values in the compact set \mathcal{K}_F .

We next tackle the stochastic convolution. To that end, we pick θ such that $\mathbf{D}((w - A)_p^\theta) \hookrightarrow E$ and $\gamma, r \in (0, \frac{1}{2})$ with $\gamma < r$, and $q > 2$ large enough such that $\theta < \gamma - \frac{1}{q} < r - \frac{1}{q}$. We then define v_n on Ω_n by

$$v_n(t) := \int_0^t (t-s)^{-\gamma} S(t-s)G_n(u_n(s))dW_H^n.$$

It was seen in the proof of [32, Proposition 4.2], that this is a well defined stochastic process in $L^q(\Omega, L^q(0, T; \mathbf{D}((w - A)_p^\theta))) \hookrightarrow L^q(\Omega; L^q(0, T; E))$ with

$$\mathbb{E}_n \|v_n(t)\|_{L^q(0, T; E)}^q \lesssim \int_0^T \mathbb{E}_n \|s \mapsto (t-s)^{-r} G_n(u(s))\|_{\gamma(L^2(0, t; H), L^p(\mathcal{O}))} dt.$$

Proceeding similarly as in the estimate for the stochastic term in Step 1, it is easy to see that

$$\mathbb{E}_n \|v_n(t)\|_{L^q(0, T; E)}^q \lesssim C(1 + \|u_0\|^q),$$

where C is a constant depending on q and T . Thus, by Chebyshev's inequality, we find a bounded subset $\mathcal{B}_G \subset L^q(0, T; E)$ such that $\mathbb{P}_n(v_n|_{[0, T]} \in \mathcal{B}_G) \geq 1 - \varepsilon/2$.

Via the stochastic Fubini theorem, it is seen that

$$\int_0^t S(t-s)G_n(u_n(s))dW_H^n(s) = R_\gamma v_n(t)$$

see [9, 32]. Hence, by compactness of R_γ , there exists a compact subset \mathcal{K}_G of $C([0, T]; E)$ such that for all $n \in \mathbb{N}$ the stochastic convolution $t \mapsto \int_0^t S(t-s)G_n(u_n(s))dW_H^n(s)$ lies in \mathcal{K}_G with probability at least $1 - \varepsilon/2$.

Putting $\mathcal{K} := S(\cdot)u_0 + \mathcal{K}_F + \mathcal{K}_G$, it follows, using that

$$u_n(t) = S(t)u_0 + \int_0^t S(t-s)F(u_n(s))ds + \int_0^t S(t-s)G_n(u_n(s))dW_H^n(s)$$

almost surely for all $t \in [0, T]$, that $\mathbb{P}_n(u_n|_{[0, T]} \in \mathcal{K}) \geq 1 - \varepsilon$. This finishes the proof of Step 2.

The existence of a measure $\mathbf{P} \in \mathcal{P}(C([0, \infty); E))$ which solves the local martingale problem for $(\mathcal{L}_{[A, F, G]}, \delta_{u_0})$ follows Lemma 3.9. \square

5.3. End of the proof. From what was done so far, it follows that the local martingale problem for $\mathcal{L}_{[A, F, G]}$ is well-posed. Thus, by Theorem 3.8, it is completely well-posed and the solutions are strong Markov processes with a common transition semigroup \mathbf{T} . By Theorem 3.5, there exist weak solutions of equation $[A, F, G]$, that is, of equation (1.2), with any specified initial distribution $\mu \in \mathcal{P}(E)$. Arguing similar as in the previous section, one sees that for any solution u the stochastic convolution exists as a stochastic integral in $\mathbf{D}((w - A)_p^\theta)$ and hence, trivially, in E . This yields that u is, in fact, a mild solution. With similar arguments as before, one also sees that for deterministic initial values $u_n \rightarrow u$ in E , it follows that \mathbf{P}_{u_n}

converges weakly to \mathbf{P}_u . Now the claimed properties of the transition semigroup \mathbf{T} follow from Theorem 2.5. The proof of Theorem 5.4 is now complete.

Acknowledgment. I would like to thank Jan van Neerven for helpful comments and for suggesting several improvements.

REFERENCES

- [1] Zdzisław Brzeźniak. Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.*, 4(1):1–45, 1995.
- [2] Zdzisław Brzeźniak and Dariusz Gątarek. Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces. *Stochastic Process. Appl.*, 84(2):187–225, 1999.
- [3] Zdzisław Brzeźniak, Bohdan Masłowski, and Jan Seidler. Stochastic nonlinear beam equations. *Probab. Theory Related Fields*, 132(1):119–149, 2005.
- [4] K Burdzy, C Mueller, and E.A. Perkins. Non-uniqueness for non-negative solutions of parabolic stochastic partial differential equations. preprint, arXiv:1008.2126v2.
- [5] Donald L. Burkholder. Martingales and singular integrals in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 233–269. North-Holland, Amsterdam, 2001.
- [6] S. Cerrai. Stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Probab. Theory Related Fields*, 125(2):271–304, 2003.
- [7] A. Chojnowska-Michalik and B. Goldys. Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces. *Probab. Theory Related Fields*, 102(3):331–356, 1995.
- [8] S. Cox and M. C. Veraar. Vector-valued decoupling and the Burkholder-Davis-Gundy inequality. to appear in, 2010.
- [9] G. Da Prato, S. Kwapień, and J. Zabczyk. Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics*, 23(1):1–23, 1987.
- [10] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [11] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.
- [12] Bálint Farkas. Perturbations of bi-continuous semigroups with applications to transition semigroups on $C_b(H)$. *Semigroup Forum*, 68(1):87–107, 2004.
- [13] D. J. H. Garling. Brownian motion and UMD-spaces. In *Probability and Banach spaces (Zaragoza, 1985)*, volume 1221 of *Lecture Notes in Math.*, pages 36–49. Springer, Berlin, 1986.
- [14] Dariusz Gątarek and Benjamin Goldys. On uniqueness in law of solutions to stochastic evolution equations in Hilbert spaces. *Stochastic Anal. Appl.*, 12(2):193–203, 1994.
- [15] Dariusz Gątarek and Benjamin Goldys. On weak solutions of stochastic equations in Hilbert spaces. *Stochastics*, 46(1-2):41–51, 1994.
- [16] Dariusz Gątarek. A note on nonlinear stochastic equations in Hilbert spaces. *Statist. Probab. Lett.*, 17(5):387–394, 1993.
- [17] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [18] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [19] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
- [20] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [21] Gottfried Köthe. *Topological vector spaces. I*. Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York, 1969.
- [22] M. C. Kunze. Perturbation of strong feller semigroups and well-posedness of semilinear stochastic equations on banach spaces. preprint, arXiv:1101.2369, 2011.
- [23] M. C. Kunze and J. M. A. M. van Neerven. Continuous dependence on the coefficients and global existence for stochastic reaction diffusion equations. preprint, arXiv:1104.4258, 2011.
- [24] Markus Kunze. Continuity and equicontinuity of semigroups on norming dual pairs. *Semigroup Forum*, 79(3):540–560, 2009.
- [25] Markus Kunze. A Pettis-type integral and applications to transition semigroups. *Czechoslovak Math. J.*, 61(2):437 – 459, 2011.

- [26] Terry R. McConnell. Decoupling and stochastic integration in UMD Banach spaces. *Probab. Math. Statist.*, 10(2):283–295, 1989.
- [27] Leonid Mytnik, Edwin Perkins, and Anja Sturm. On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients. *Ann. Probab.*, 34(5):1910–1959, 2006.
- [28] J. M. A. M. van Neerven. *The adjoint of a semigroup of linear operators*, volume 1529 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.
- [29] J. M. A. M. van Neerven. γ -Radonifying operators: a survey. In *AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis*, pages 1–62. 2010.
- [30] J. M. A. M. van Neerven and M. C. Veraar. On the stochastic Fubini theorem in infinite dimensions. In *Stochastic partial differential equations and applications—VII*, volume 245 of *Lect. Notes Pure Appl. Math.*, pages 323–336. Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [31] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis. Stochastic integration in UMD Banach spaces. *Ann. Probab.*, 35(4):1438–1478, 2007.
- [32] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis. Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal.*, 255(4):940–993, 2008.
- [33] J. M. A. M. van Neerven and L. Weis. Stochastic integration of functions with values in a Banach space. *Studia Math.*, 166(2):131–170, 2005.
- [34] Martin Ondreját. Uniqueness for stochastic evolution equations in Banach spaces. *Dissertationes Math. (Rozprawy Mat.)*, 426:63, 2004.
- [35] Martin Ondreját. Brownian representations of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces. *Czechoslovak Math. J.*, 55(130)(4):1003–1039, 2005.
- [36] Martin Ondreját. Integral representations of cylindrical local martingales in every separable Banach space. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 10(3):365–379, 2007.
- [37] E. M. Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.
- [38] K. R. Parthasarathy. *Probability measures on metric spaces*. AMS Chelsea Publishing, Providence, RI, 2005. Reprint of the 1967 original.
- [39] R. Schnaubelt and M. C. Veraar. Structurally damped plate and wave equations with random point force in arbitrary space dimensions. to appear in *Differential and Integral equations*, 2010.
- [40] F. Dennis Sentes. Bounded continuous functions on a completely regular space. *Trans. Amer. Math. Soc.*, 168:311–336, 1972.
- [41] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes with continuous coefficients. I and II. *Comm. Pure Appl. Math.*, 22:345–400 and 479–530, 1969.
- [42] M. Viot. *Solutions faibles d’équations aux dérivées partielles non linéaires*. PhD thesis, Université Paris VI, 1976.
- [43] Bin Xie. On pathwise uniqueness of stochastic evolution equations in Hilbert spaces. *J. Math. Anal. Appl.*, 344(1):204–216, 2008.
- [44] Toshio Yamada and Shinzo Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, 11:155–167, 1971.
- [45] Lorenzo Zambotti. An analytic approach to existence and uniqueness for martingale problems in infinite dimensions. *Probab. Theory Related Fields*, 118(2):147–168, 2000.
- [46] J. Zimmerschied. *Über eine Faktorisierungsmethode für stochastische Evolutionsgleichungen in Banachräumen*. PhD thesis, Universität Karlsruhe, 2006.

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